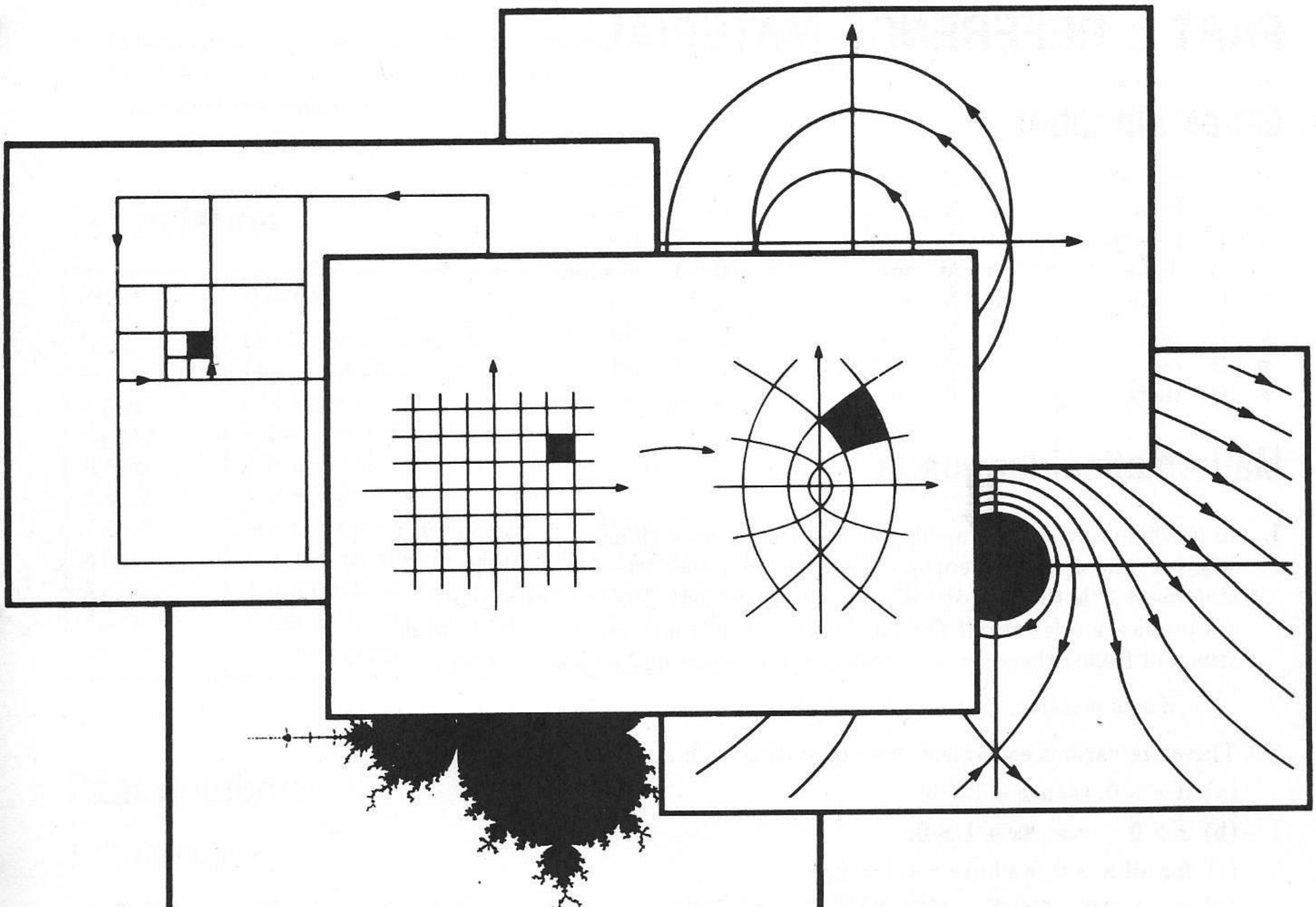
M337 HB1
Mathematics: A Third Level Course

COMPLEX ANALYSIS

HANDBOOK ONE



MPLEX ANALYSIS COMPLEX ANALYSIS COMPL

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CONTENTS

PART I: REFERENCE MATERIAL

Greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ε	E	epsilon	ν	N	nu	ϕ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

Mathematical language

1. In mathematics we commonly use **implications** such as 'if P then Q', where P and Q are statements which can be either true or false. The statement P is the **hypothesis** and the statement Q is the **conclusion**. An implication is **true** if the conclusion is true whenever the hypothesis is true and **false** otherwise. For example, the following implication is true:

if x is positive, then $x + 1$ is positive.

There are various equivalent ways of stating such an implication:

(a) if $x > 0$, then $x + 1 > 0$;

(b) $x > 0 \quad \Longrightarrow \quad x + 1 > 0$;

(c) for all $x > 0$, we have $x + 1 > 0$;

(d) $x + 1 > 0$, for all $x > 0$;

(e) $x + 1 > 0$, whenever $x > 0$;

(f) for $x + 1$ to be positive, it is sufficient that x be positive.

2. The **converse** of an implication is obtained by exchanging the hypothesis and the conclusion. For example, the converse of the (true) implication

if $x > 0$, then $x + 1 > 0$

is

if $x + 1 > 0$, then $x > 0$,

which is false (try $x = 0$).

3. An **equivalence** consists of an implication 'if P then Q' *and* its converse 'if Q then P'. The equivalence is true if both these implications are true. For example, the following equivalence is true:

$x > 0$ is equivalent to $2x > 0$.

It could alternatively be stated as follows:

$x > 0 \iff 2x > 0$;

$x > 0$ if and only if $2x > 0$;

$x > 0$ is necessary and sufficient for $2x > 0$.

4. There are three ways of proving implications:
   (a) **direct proof**: we begin by assuming that the hypothesis is true and then argue directly to show that the conclusion is true;
   (b) **proof by contraposition**: we begin by assuming that the conclusion is false and then argue directly to show that the hypothesis is false;
   (c) **proof by contradiction**: we begin by assuming that the hypothesis is true *and* that the conclusion is false, and then argue from both to obtain a contradiction.

   Where possible, types (a) or (b) are preferable, since they establish a direct link between hypothesis and conclusion, but type (c) is often convenient and sometimes essential.

Set notation

Notation	Meaning
$\{x, y, \ldots, z\}$	The set of elements listed in $\{\ldots\}$
$\{x : \ldots\}$	The set of all x such that ... holds
$x \in A$	x belongs to A
$x \notin A$	x does not belong to A
$A \subseteq B$	A is a subset of B: each element of A belongs to B
$A = B$	A is equal to B: $A \subseteq B$ and $B \subseteq A$
$A \subset B$	A is a proper subset of B: $A \subseteq B$ but $A \neq B$
$A \cup B$	A union B: the set of all elements which belong to A or B (or both)
$A \cap B$	A intersection B: the set of all elements which belong to both A and B
$A - B$	A minus B: the set of all elements of A which do not belong to B
$\varnothing$	The empty set

Some texts use $A \subset B$ to mean $A \subseteq B$.

Real numbers

1. Definitions

A **real number** is a number which can be represented by a decimal of the form

$$\pm a_0.a_1a_2a_3\ldots,$$

where a_0 is a non-negative integer and a_1, a_2, a_3, ... are digits. **Rational numbers** (ratios of integers) are represented by recurring decimals and **irrational numbers** are represented by non-recurring decimals. Real numbers are often represented by points on a line, called the **real line**.

2. Some important subsets of the real numbers

Symbol	Subset
$\mathbb{N}$	The set of all natural numbers: $1, 2, 3, \ldots$
$\mathbb{Z}$	The set of all integers: $0, \pm 1, \pm 2, \pm 3, \ldots$
$\mathbb{Q}$	The set of all rational numbers (numbers of the form p/q, where $p, q \in \mathbb{Z}, q \neq 0$)
$\mathbb{R}$	The set of all real numbers
$]a, b[$	$\{x : a < x < b\}$, the open interval from a to b
$[a, b]$	$\{x : a \leq x \leq b\}$, the closed interval from a to b
$]a, b]$	$\{x : a < x \leq b\}$
$[a, \infty[$	$\{x : x \geq a\}$
$]a, \infty[$	$\{x : x > a\}$
$]-\infty, a[$	$\{x : x < a\}$

3. Upper and lower bounds

Suppose that A is a non-empty subset of $\mathbb{R}$. Then A is **bounded above** if there is a real number M such that

$$x \leq M, \quad \text{for all } x \in A.$$

The number M is called an **upper bound** of A. Clearly, any number bigger than M is also an upper bound of A. A **lower bound** of A is defined similarly.

Among all upper bounds of A, the smallest (which always exists if A is bounded above) is called the **least upper bound** of A, or the **supremum** of A, written $\mathbf{sup}\, \boldsymbol{A}$. The **greatest lower bound**, or the **infimum**, of A, written $\mathbf{inf}\, \boldsymbol{A}$, is defined similarly.

If A has infinitely many elements, then $\sup A$ and $\inf A$ may or may not belong to A. In contrast, if A has finitely many elements, then $\sup A$ and $\inf A$ are the largest and the smallest elements of A, respectively, and therefore are members of A.

If $\sup A$ belongs to A, then we may use the alternative notation $\mathbf{max}\, \boldsymbol{A}$ for $\sup A$. Similarly, if $\inf A$ belongs to A, then we may denote it by $\mathbf{min}\, \boldsymbol{A}$.

Example If $A = [-1, 1[$, then

$$\sup A = 1 \quad \text{and} \quad \inf A = \min A = -1.$$

If $B = \{-1, 0, 1, 2\}$, then

$$\sup B = \max B = 2 \quad \text{and} \quad \inf B = \min B = -1.$$

4. Inequalities

Rules for rearranging inequalities

(a) $a < b \iff b - a > 0$.

(b) $a < b \iff a + c < b + c$.

(c) If $c > 0$, then

$$a < b \iff ac < bc;$$

if $c < 0$, then

$$a < b \iff ac > bc.$$

(d) If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

(e) If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

Corresponding versions of these inequalities exist with *strict* inequalities replaced by *weak* inequalities; that is, with $a < b$ replaced by $a \leq b$.

The **solution set** of an inequality involving an unknown real number x is the set of values of x for which the inequality holds.

Rules for deducing new inequalities from given ones

(a) For all a, b, c in $\mathbb{R}$,

$$a < b \text{ and } b < c \quad \Longrightarrow \quad a < c.$$

Transitive Rule

(b) For all a, b, c, d in $\mathbb{R}$, if $a < b$ and $c < d$, then

$$a + c < b + d;$$

Sum Rule

$$ac < bd \quad (\text{provided that } a, c \geq 0).$$

Product Rule

Modulus

If $x \in \mathbb{R}$, then the **modulus**, or **absolute value**, of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Thus $|x|$ is the distance from the origin to x, and so

(a) $|x| < a \quad \Longleftrightarrow \quad -a < x < a;$

(b) $|x| > a \quad \Longleftrightarrow \quad x > a \text{ or } x < -a;$

(c) the distance on the real line from a to b is $|a - b|$.

5. The Principle of Mathematical Induction

Suppose that $P(n)$, $n = 1, 2, 3, \ldots$, is a sequence of propositions such that

(a) $P(1)$ is true, and

(b) whenever $P(k)$ is true, $P(k + 1)$ is also true.

Then $P(n)$ is true, for $n = 1, 2, \ldots$.

Example We wish to show that $2^n \geq n$, for $n = 1, 2, \ldots$.

Let $P(n)$ be the proposition $2^n \geq n$.

(a) $P(1)$ is true, since $2^1 \geq 1$.

(b) Suppose that $P(k)$ is true; that is, $2^k \geq k$. Then

$$2^{k+1} = 2 \cdot 2^k \geq 2k = k + k \geq k + 1,$$

so that $P(k + 1)$ is true.

Thus $2^n \geq n$, for $n = 1, 2, \ldots$, by the Principle of Mathematical Induction.

Real functions

1. Definitions

A **real function** f is defined by specifying

(a) two subsets A and B of $\mathbb{R}$, and

(b) a rule which associates with each $x \in A$ a unique $y \in B$.

The sets A and B are called the **domain** and the **codomain** of f, respectively. We write

$$f : A \longrightarrow B.$$

If $f : A \longrightarrow B$ and $x \in A$, then $f(x)$ is called the **image of x under f**, or the **value of f at x**. The **image** of the function f is

$$f(A) = \{f(x) : x \in A\}.$$

2. Standard functions

Type	Rule	Domain
Polynomial functions	$a_0 + a_1x + \cdots + a_nx^n$	$\mathbb{R}$
Rational functions	$p(x)/q(x)$, p and q polynomial functions (q not the zero function)	$\mathbb{R} - \{x : q(x) = 0\}$
Trigonometric functions	$\sin x$ $\cos x$ $\tan x$	$\mathbb{R}$ $\mathbb{R}$ $\mathbb{R} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$
Exponential functions	e^x a^x, where $a > 0$	$\mathbb{R}$ $\mathbb{R}$
Natural log function	$\log_e x$	$\{x : x > 0\}$
Hyperbolic functions	$\sinh x$ $\cosh x$ $\tanh x$	$\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$

Remarks

(a) Definition of e^x:

$$e^x = \sum_{n=0}^{\infty} x^n/n! = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

(b) Definition of $\log_e x$:

$$x = e^y \implies y = \log_e x$$

(c) Definition of a^x (for $a > 0$):

$$a^x = \exp(x \log_e a)$$

(d) Index laws:

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad \sqrt[n]{a^m} = a^{m/n}, \quad a^{-x} = 1/a^x$$

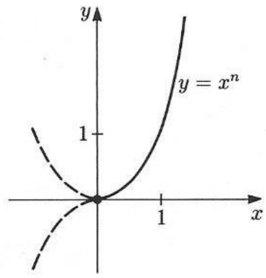
(e) Logarithmic identities:

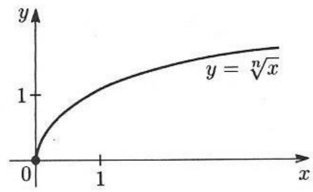
$$\log_e xy = \log_e x + \log_e y, \quad \log_e(1/x) = -\log_e x$$

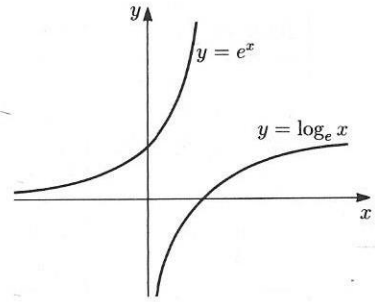
3. Trigonometric and hyperbolic identities

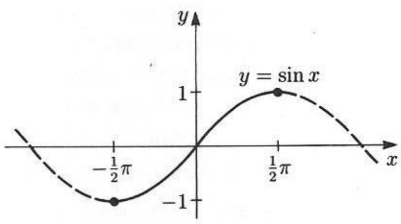
$\cos^2 x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $\cot^2 x + 1 = \operatorname{cosec}^2 x$	$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\sinh 2x = 2 \sinh x \cosh x$ $\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$	$\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$ $\tanh(-x) = -\tanh x$

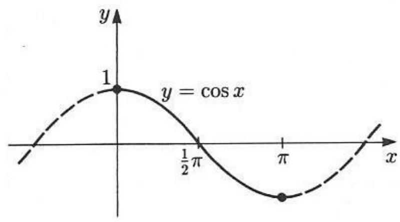
4. Graphs of standard functions and their inverses

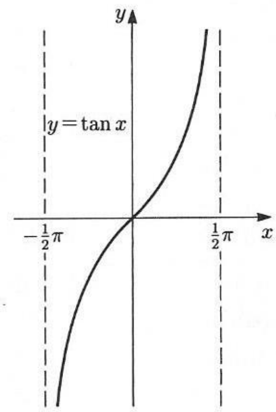


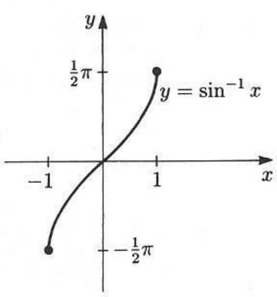


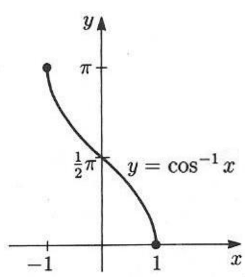


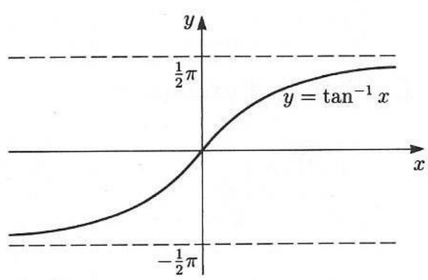


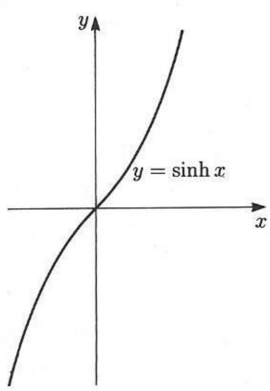


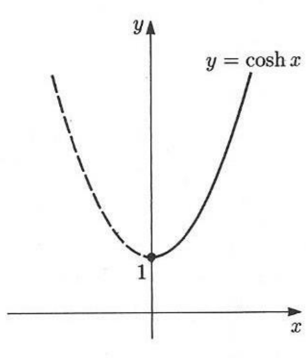


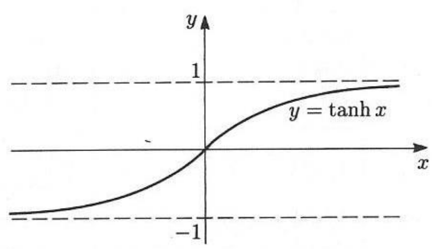


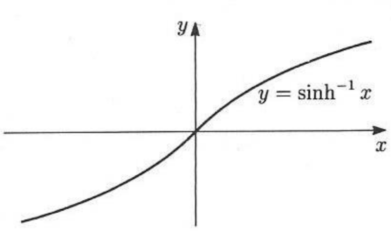


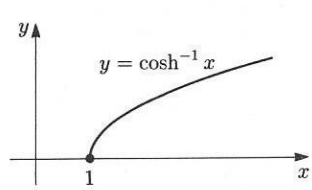


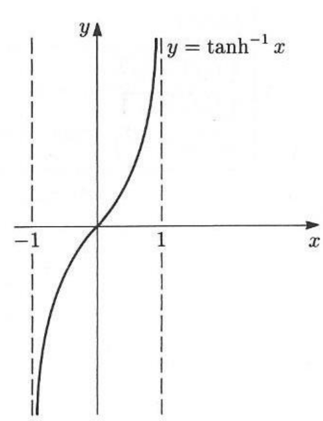












5. Standard derivatives

Rule of f	Rule of f'	Domain of f'
k	0	$\mathbb{R}$
$x^n,\ n \in \mathbb{Z} - \{0\}$	nx^{n-1}	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 0$
$x^\alpha,\ \alpha \in \mathbb{R} - \{0\}$	$\alpha x^{\alpha-1}$	$]0, \infty[$
e^x	e^x	$\mathbb{R}$
$a^x,\ a > 0$	$a^x \log_e a$	$\mathbb{R}$
$\log_e x$	$1/x$	$]0, \infty[$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sec x$	$\sec x \tan x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sin^{-1} x$	$1/\sqrt{1 - x^2}$	$]-1, 1[$
$\cos^{-1} x$	$-1/\sqrt{1 - x^2}$	$]-1, 1[$
$\tan^{-1} x$	$1/(1 + x^2)$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\operatorname{sech}^2 x$	$\mathbb{R}$
$\sinh^{-1} x$	$1/\sqrt{1 + x^2}$	$\mathbb{R}$
$\cosh^{-1} x$	$1/\sqrt{x^2 - 1}$	$]1, \infty[$
$\tanh^{-1} x$	$1/(1 - x^2)$	$]-1, 1[$

6. Standard primitives

Rule of f	Rule of F $(F' = f)$	Domain of F
$x^n,\ n \in \mathbb{Z} - \{-1, 0\}$	$x^{n+1}/(n + 1)$	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 1$
$x^\alpha,\ \alpha \in \mathbb{R} - \{-1, 0\}$	$x^{\alpha+1}/(\alpha + 1)$	$]0, \infty[$
$1/x$	$\log_e \lvert x \rvert$	$\mathbb{R} - \{0\}$
e^x	e^x	$\mathbb{R}$
$a^x,\ a > 0$	$a^x / \log_e a$	$\mathbb{R}$
$\log_e x$	$x \log_e x - x$	$]0, \infty[$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\tan x$	$\log_e \lvert \sec x \rvert$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$1/\sqrt{1 - x^2}$	$\begin{cases} \sin^{-1} x \\ -\cos^{-1} x \end{cases}$	$]-1, 1[$ $]-1, 1[$
$1/(1 + x^2)$	$\tan^{-1} x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\log_e(\cosh x)$	$\mathbb{R}$
$1/\sqrt{1 + x^2}$	$\sinh^{-1} x$	$\mathbb{R}$
$1/\sqrt{x^2 - 1}$	$\cosh^{-1} x$	$]1, \infty[$
$1/(1 - x^2)$	$\tanh^{-1} x$	$]-1, 1[$

PART II: UNIT SUMMARIES

Unit A1 Complex Numbers

Section 1: Introducing Complex Numbers

1. A **complex number** z is an expression of the form $x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol with the property that $i^2 = -1$. We write $z = x + iy$ or, equivalently, $z = x + yi$, and say that z is expressed in **Cartesian form**; x is called the **real part** of z and y the **imaginary part** of z, written $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.
Two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal.
The set of all complex numbers is denoted by $\mathbb{C}$.

2. The binary operations of **addition**, **subtraction** and **multiplication** of complex numbers are denoted by the same symbols as for real numbers and are performed by treating the complex numbers as real expressions involving an algebraic symbol i with the property that $i^2 = -1$.

3. The **negative**, $-z$, of a complex number $z = x + iy$ is

$$-z = (-x) + i(-y),$$

usually written $-z = -x - iy$.

4. The **reciprocal**, $1/z$ (or z^{-1}), of a non-zero complex number $z = x + iy$ is

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

5. The **quotient**, z_1/z_2, of a complex number z_1 by a non-zero complex number z_2 is

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right).$$

6. Strategy for obtaining a quotient

To obtain the quotient

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \quad \text{where } x_2 + iy_2 \neq 0,$$

in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

7. The **complex conjugate**, $\overline{z}$, of a complex number $z = x + iy$ is

$$\overline{z} = x - iy.$$

8. Properties of the complex conjugate

(a) If z is a complex number, then

(i) $z + \overline{z} = 2\operatorname{Re} z$;

(ii) $z - \overline{z} = 2i\operatorname{Im} z$;

(iii) $\overline{(\overline{z})} = z$.

(b) If z_1 and z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(iii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;

(iv) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$, where $z_2 \neq 0$.

9. Arithmetic in $\mathbb{C}$

Addition	*Multiplication*
A1 For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 \in \mathbb{C}$.	M1 For all z_1, z_2 in $\mathbb{C}$, $z_1 z_2 \in \mathbb{C}$.
A2 For all z in $\mathbb{C}$, $z + 0 = 0 + z = z$.	M2 For all z in $\mathbb{C}$, $z1 = 1z = z$.
A3 For all z in $\mathbb{C}$, $z + (-z) = (-z) + z = 0$.	M3 For all non-zero z in $\mathbb{C}$, $zz^{-1} = z^{-1}z = 1$.
A4 For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.	M4 For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1 z_2)z_3 = z_1(z_2 z_3)$.
A5 For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 = z_2 + z_1$.	M5 For all z_1, z_2 in $\mathbb{C}$, $z_1 z_2 = z_2 z_1$.

D For all z_1, z_2, z_3 in $\mathbb{C}$, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

10. Binomial Theorem

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$(1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + z^n.$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k = z_1^n + nz_1^{n-1}z_2 + \frac{n(n-1)}{2!}z_1^{n-2}z_2^2 + \cdots + z_2^n.$$

11. Geometric Series Identity

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$1 - z^n = (1 - z)\left(1 + z + z^2 + \cdots + z^{n-1}\right).$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$z_1^n - z_2^n = (z_1 - z_2)\left(z_1^{n-1} + z_1^{n-2}z_2 + z_1^{n-3}z_2^2 + \cdots + z_2^{n-1}\right).$$

Section 2: The Complex Plane

1. The complex number $z = x + iy$ is represented by the ordered pair (x, y) in $\mathbb{R}^2$, called the **complex plane.**

Complex numbers can be thought of as based vectors; $z = x + iy$ corresponds to the vector from $(0,0)$ to (x,y). The sum and difference of two complex numbers z_1 and z_2 satisfy the parallelogram law for vectors.

2. The **modulus,** or **absolute value,** of $z = x + iy$ is the distance from 0 to z; it is denoted by $|z|$. Thus

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

3. $|z_1 - z_2|$ is the distance from z_1 to z_2.
$|z_1 + z_2|$ is the distance from z_1 to $-z_2$.

4. **Properties of the modulus**

(a) $|z| \geq 0$, with equality if and only if $z = 0$;

(b) $|\overline{z}| = |z|$ and $|-z| = |z|$;

(c) $|z|^2 = z\overline{z}$;

(d) $|z_1 - z_2| = |z_2 - z_1|$;

(e) $|z_1 z_2| = |z_1||z_2|$
and
$|z_1/z_2| = |z_1|/|z_2|$, where $z_2 \neq 0$.

5. An **argument** of $z = x + iy$, $z \neq 0$, is an angle θ (measured in radians) such that

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

No argument is assigned to $z = 0$.

Each $z \neq 0$ has infinitely many arguments. If θ is an argument of z, then the arguments of z are given by $\theta + 2n\pi$, $n \in \mathbb{Z}$.

6. The ordered pair (r, θ), where r is the modulus of a non-zero complex number z and θ is an argument of z, are called **polar coordinates** of z. The expression

$$z = r(\cos\theta + i\sin\theta)$$

is called a representation of z in **polar form.**

7. The **principal argument** of a non-zero complex number z is the unique argument θ of z satisfying $-\pi < \theta \leq \pi$; it is denoted by

$$\theta = \operatorname{Arg} z.$$

8. **Strategy for determining principal arguments**

To determine the principal argument θ of a non-zero complex number $z = x + iy$.

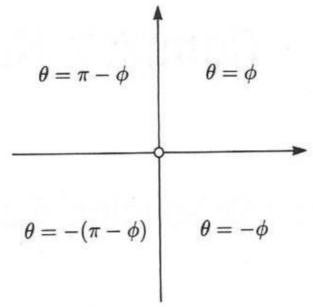
Case (i) If z lies on one of the axes, then θ is evident.

Case (ii) If z does not lie on one of the axes, then

(a) decide in which quadrant z lies (by plotting if necessary) and then calculate the angle

$$\phi = \tan^{-1}(|y|/|x|);$$

(b) obtain θ in terms of ϕ by using the appropriate formula in the following diagram.



9. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

The geometric effect on z_1 of multiplying it by z_2 is to scale z_1 by the factor $|z_2|$ and rotate it about 0 through the angle $\operatorname{Arg} z_2$ (this rotation is anticlockwise if $\operatorname{Arg} z_2 > 0$, clockwise if $\operatorname{Arg} z_2 < 0$).

10. If z_1 and z_2 are (non-zero) complex numbers, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi,$$

where n is -1, 0 or 1 according as $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $]-\pi, \pi]$ or is less than or equal to $-\pi$.

11. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

The geometric effect on z_1 of dividing it by z_2 is to scale z_1 by the factor $1/|z_2|$ and rotate it about 0 through the angle $\operatorname{Arg} z_2$ (this rotation is clockwise if $\operatorname{Arg} z_2 > 0$, anticlockwise if $\operatorname{Arg} z_2 < 0$).

12. If z is non-zero and $z = r(\cos\theta + i\sin\theta)$, then

$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)).$$

13. **de Moivre's Theorem**

If n is an integer and θ is a real number, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Section 3: Solving Equations with Complex Numbers

1. Suppose that w is a non-zero complex number and $n \geq 2$. Each solution of $z^n = w$ is called an **nth root** of w.

2. Let

$$w = \rho(\cos\phi + i\sin\phi)$$

be a non-zero complex number in polar form. Then w has exactly n nth roots, given by

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$
$$k = 0, 1, \ldots, n-1.$$

These n nth roots form the vertices of an n-sided regular polygon inscribed in a circle of radius $\rho^{1/n}$ centred at 0.

3. If $w = \rho(\cos\phi + i\sin\phi)$, where ϕ is the principal argument of w, then

$$z_0 = \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)$$

is called the **principal nth root** of w, denoted by $\sqrt[n]{w}$ or $w^{1/n}$.

4. By an **nth root of unity** we mean an nth root of 1. These n roots are given by

$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

5. **Strategy for finding nth roots**

To find the n nth roots $z_0, z_1, \ldots, z_{n-1}$ of a non-zero complex number w:

(a) express w in polar form, with modulus ρ and argument ϕ;

(b) substitute the values of ρ and ϕ in the formula

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + \sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$
$$k = 0, 1, \ldots, n-1;$$

(c) if required, convert the roots to Cartesian form.

6. The two roots of $az^2 + bz + c = 0$, where a, b, c are in $\mathbb{C}$ and $a \neq 0$, are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In certain cases a polynomial equation of degree n can be solved by substitution to obtain a quadratic equation.

Section 4: Sets of Complex Numbers

1. Inequalities between complex numbers have no meaning, unless the complex numbers are real. Since $|z|$, $\operatorname{Re} z$, $\operatorname{Im} z$ and $\operatorname{Arg} z$ are real, we can form inequalities involving them.

2. The following subsets of $\mathbb{C}$ are commonly used.

(a) Open half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z > c\}$,
closed half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z \geq c\}$,
where $a, b, c \in \mathbb{R}$ and a, b are not both zero.

(b) Open disc $\{z : |z - \alpha| < r\}$,
closed disc $\{z : |z - \alpha| \leq r\}$,
where $r > 0$.

(c) Open annulus $\{z : r_1 < |z - \alpha| < r_2\}$,
closed annulus $\{z : r_1 \leq |z - \alpha| \leq r_2\}$,
where $r_2 \geq r_1 \geq 0$.

(d) Ray (half-line) $\{z : \operatorname{Arg} z = \theta\}$,
where $\theta \in]-\pi, \pi]$.

(e) Open sector $\{z : a < \operatorname{Arg}(z - \alpha) < b\}$,
where $\alpha \in \mathbb{C}$ and $-\pi \leq a < b \leq \pi$.

A special case is the cut plane

$$\{z : |\operatorname{Arg} z| < \pi\} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$$

Section 5: Proving Inequalities

1. $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

2. **Triangle Inequality**

If z_1, $z_2 \in \mathbb{C}$, then

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$ (usual form);

(b) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (backwards form).

3. If z, z_1, z_2, …, $z_n \in \mathbb{C}$, then

(a) $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$;

(b) $|z_1 - z_2| \leq |z_1| + |z_2|$;

(c) $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$;

(d) $|z_1 \pm z_2 \pm \cdots \pm z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$;

(e) $|z_1 \pm z_2 \pm \cdots \pm z_n| \geq |z_1| - |z_2| - \cdots - |z_n|$.

Unit A2 Complex Functions

Section 1: What is a Complex Function?

1. A **complex function** f is defined by specifying

(a) two sets A and B in the complex plane $\mathbb{C}$, and

(b) a rule which associates with each number z in A a unique number w in B; we write $w = f(z)$.

The set A is called the **domain** of the function f and B is called the **codomain** of f. The number w is called the **image of z under f**, or the **value of f at z**.

2. **Convention** When a function f is specified *just* by its rule, it is to be understood that the domain of f is the set of all complex numbers to which the rule is applicable, and the codomain of f is $\mathbb{C}$.

3. Given a function $f : A \longrightarrow B$, the **image of f** is

$$f(A) = \{f(z) : z \in A\}.$$

If $f(A) = B$, then f is called **onto**.

4. If $f(A) \subseteq \mathbb{R}$, then f is called **real-valued**.
If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, then f is a **real function**.

5. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be functions. Then the **sum** $f + g$ is the function with domain $A \cap B$ and rule

$$(f+g)(z) = f(z) + g(z);$$

the **multiple** λf, where $\lambda \in \mathbb{C}$, is the function with domain A and rule

$$(\lambda f)(z) = \lambda f(z);$$

the **product** fg is the function with domain $A \cap B$ and rule

$$(fg)(z) = f(z)g(z);$$

the **quotient** f/g is the function with domain $A \cap B - \{z : g(z) = 0\}$ and rule

$$(f/g)(z) = f(z)/g(z).$$

6. A **polynomial function** of degree n is defined by

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

A **rational function** r is the quotient of two polynomial functions p and q: $r(z) = \dfrac{p(z)}{q(z)}$.

7. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be complex functions. Then the **composite function** $g \circ f$ has domain

$$\{z \in A : f(z) \in B\}$$

and rule

$$(g \circ f)(z) = g(f(z)).$$

8. The function $f : A \longrightarrow B$ is **one-one** if the images under f of distinct points in A are also distinct; that is,

$$\text{if } z_1, z_2 \in A \text{ and } z_1 \neq z_2, \quad \text{then } f(z_1) \neq f(z_2).$$

Equivalently, if $w \in f(A)$ then there is a *unique* $z \in A$ such that $f(z) = w$.

A function which is not one-one is **many-one**.

9. Let $f : A \longrightarrow B$ be a one-one function. Then the **inverse function**, f^{-1}, of f has domain $f(A)$ and rule

$$f^{-1}(w) = z, \qquad \text{where } w = f(z).$$

10. **Strategy for proving that a function f has an inverse function**

EITHER prove that f is one-one directly by showing that

$$\text{if } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2)$$

OR determine the image $f(A)$ and show that for each $w \in f(A)$ there is a unique $z \in A$ such that

$$f(z) = w.$$

11. Reducing the domain of a function (but leaving the rule unchanged) gives a **restriction** of the function.

Section 2: Special Types of Complex Function

1. The functions $\operatorname{Re} f : z \longmapsto \operatorname{Re}(f(z))$ and $\operatorname{Im} f : z \longmapsto \operatorname{Im}(f(z))$ are called the **real** and the **imaginary parts** of f. $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions with the same domain as f.

2. A **path** is a subset Γ of $\mathbb{C}$ which is the image of an associated continuous function $\gamma : I \longrightarrow \mathbb{C}$, where I is a real interval.

The function γ is called a **parametrization**. If

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$

where ϕ and ψ are real functions, then the equations

$$x = \phi(t), \quad y = \psi(t) \quad (t \in I),$$

are called **parametric equations**.

If I is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of Γ, respectively.

It may be possible to obtain the equation of Γ in terms of x and y alone by eliminating t from the equations $x = \phi(t)$ and $y = \psi(t)$.

3. Standard parametrizations

Set	Standard parametrization
Line through points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in \mathbb{R})$
Line segment between points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in [0,1])$
Circle with centre α, radius r: $\|z-\alpha\| = r$.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [0, 2\pi])$
Arc of circle with centre α, radius r.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [t_1, t_2])$
Ellipse in standard form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.	$\gamma(t) = a\cos t + ib\sin t$ $(t \in [0, 2\pi])$
Parabola in standard form: $y^2 = 4ax$.	$\gamma(t) = at^2 + 2iat$ $(t \in \mathbb{R})$
Right half of hyperbola in standard form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.	$\gamma(t) = a\cosh t + ib\sinh t$ $(t \in \mathbb{R})$

4. Given a function $f : A \longrightarrow B$ and a subset S of A, the **image under f of S** is

$$f(S) = \{f(z) : z \in S\}.$$

5. If f is a continuous function and Γ is a path in the domain of f, with parametrization γ, then $f(\Gamma)$ is called the **image path**; $f(\Gamma)$ has parametrization $f \circ \gamma$.

6. Strategy for determining an image path

Let the parametrization of the path Γ be

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I).$$

Then the image path $f(\Gamma)$ is found

EITHER

by using the geometric properties of the continuous function f

OR

by substituting $x = \phi(t)$, $y = \psi(t)$ into the equation

$$u + iv = f(x + iy)$$

and then, by equating real parts and imaginary parts, obtaining expressions for u and v in terms of t. (These expressions are the parametric equations of the image path $f(\Gamma)$, associated with the parametrization $f \circ \gamma$.)

Section 3: Images of Grids

A **Cartesian grid** consists of lines of the form $x = a$ and $y = b$, usually evenly spaced in both directions.

A **polar grid** consists of circles centred at 0 and rays emerging from 0. Each of the circles has an equation of the form $r = a$, where a is a positive constant, and each of the rays has an equation of the form $\theta = b$, where b is a constant in the interval $]-\pi, \pi]$.

Using the Strategy in Section 2 for determining an image path $f(\Gamma)$, we can obtain the images of Cartesian and polar grids.

Section 4: Exponential, Trigonometric and Hyperbolic Functions

1. For all $z = x + iy$ in $\mathbb{C}$,

$$e^z = e^x(\cos y + i\sin y).$$

The function

$$z \longmapsto e^z \quad (z \in \mathbb{C})$$

is called the **exponential function** and is denoted by exp.

If z is real, $z = x + 0i$, then

$$e^z = e^x(\cos 0 + i\sin 0) = e^x.$$

If z is imaginary, $z = 0 + iy$, then

$$e^{iy} = \cos y + i\sin y.$$

2. Exponential identities

(a) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(b) $|e^z| = e^{\operatorname{Re} z}$

(c) $e^{-z} = 1/e^z$

(d) $e^{z+2\pi i} = e^z$

3. The geometric nature of exp

(a) For all $n \in \mathbb{Z}$, $e^{z+2n\pi i} = e^z$, so each of the points $z + 2n\pi i$, $n \in \mathbb{Z}$, has the same image.

(b) The line $x = a$ is mapped to the path with parametric equations

$$u = e^a\cos t, \quad v = e^a \sin t \quad (t \in \mathbb{R}).$$

This is the circle with radius e^a and centre 0.

(c) The line $y = b$ is mapped to the path with parametric equations

$$u = e^t\cos b, \quad v = e^t\sin b \quad (t \in \mathbb{R}).$$

This is the ray from 0 (excluded) through $\cos b + i\sin b$.

(d) The strip $\{x + iy : -\pi < y \le \pi\}$ is mapped to $\mathbb{C} - \{0\}$.

4. **Trigonometric functions**

The functions cos, sin, tan, sec, cot and cosec are defined as follows:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$
$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z},$$
$$\cot z = \frac{\cos z}{\sin z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

Both cos and sin have domain $\mathbb{C}$.

Since $\{z : \cos z = 0\} = \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$, both tan and sec have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$.

Since $\{z : \sin z = 0\} = \{n\pi : n \in \mathbb{Z}\}$, both cot and cosec have domain $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$.

5. All the identities satisfied by the real trigonometric functions also hold for the complex trigonometric functions.

6. **Hyperbolic functions**

The functions cosh, sinh, tanh, sech, coth and cosech are defined as follows:

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right),$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z},$$
$$\coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}.$$

Both cosh and sinh have domain $\mathbb{C}$.

Since $\{z : \cosh z = 0\} = \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$, both tanh and sech have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$.

Since $\{z : \sinh z = 0\} = \{n\pi i : n \in \mathbb{Z}\}$, both coth and cosech have domain $\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}$.

7. For all z in $\mathbb{C}$,

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z.$$

8. All the identities satisfied by the real hyperbolic functions also hold for the complex hyperbolic functions.

Section 5: Logarithms and Powers

1. For $z \in \mathbb{C} - \{0\}$, the **principal logarithm** of z is

$$\operatorname{Log} z = \log_e |z| + i \operatorname{Arg} z.$$

The corresponding **principal logarithm function** is called Log. It is the inverse of

$$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \le \pi\}),$$

and satisfies

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$

and

$$\operatorname{Log}(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \le \pi\}.$$

2. **Logarithmic identities**

(a) $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$,
if $\operatorname{Arg} z_1, \operatorname{Arg} z_2 \in]-\frac{1}{2}\pi, \frac{1}{2}\pi]$;

(b) $\operatorname{Log}(1/z) = -\operatorname{Log} z$, if $\operatorname{Arg} z \in]-\pi, \pi[$.

Part (a) holds in the following form for any values in the domain, $\mathbb{C} - \{0\}$, of Log:

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2n\pi i,$$

where n is -1, 0 or 1 according as $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $]-\pi, \pi]$, or is less than or equal to $-\pi$.

(c) For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the **principal αth power** of z is

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

This definition agrees with the usual meaning of z^α, for $\alpha = n$ or $\alpha = 1/n$, where $n \in \mathbb{N}$.

The function $z \longmapsto z^\alpha$ is called the **principal αth power function**.

Unit A3 Continuity

Section 1: Sequences

1. A **(complex) sequence** is an unending list of complex numbers

$z_1, z_2, z_3, \ldots.$

The complex number z_n is called the **nth term of the sequence**, and the sequence is denoted by $\{z_n\}$.

2. The sequence $\{z_n\}$ is **convergent with limit α**, or **converges to α**, or **tends to α**, if for each positive number ε, there is an integer N such that

$|z_n - \alpha| < \varepsilon, \quad$ for all $n > N$.

If $\{z_n\}$ converges to α, then we write

EITHER $\lim\limits_{n\to\infty} z_n = \alpha$

OR $\quad z_n \to \alpha$ as $n \to \infty$.

If the limit α is 0, then $\{z_n\}$ is called a **null sequence.**

3. (a) The sequence $\{z_n\}$ converges to α if and only if the sequence $\{z_n - \alpha\}$ is null.

(b) If a sequence $\{z_n\}$ is convergent, then it has a *unique* limit.

(c) A **constant sequence**

$z_n = \alpha, \quad n = 1, 2, \ldots,$

is convergent with limit α.

(d) If a given sequence converges to α, then this remains true if we *add, delete, or alter a finite number of terms.*

4. Squeeze Rule

If $\{a_n\}$ is a real null sequence of non-negative terms, and if

$|z_n| \le a_n, \quad$ for $n = 1, 2, \ldots,$

then $\{z_n\}$ is a null sequence.

5. Basic null sequences

The following sequences are null:

(a) $\{1/n^p\}$, for $p > 0$;

(b) $\{\alpha^n\}$, for $|\alpha| < 1$.

6. Combination Rules

If $\lim\limits_{n\to\infty} z_n = \alpha$ and $\lim\limits_{n\to\infty} w_n = \beta$, then

Sum Rule $\quad \lim\limits_{n\to\infty} (z_n + w_n) = \alpha + \beta$;

Multiple Rule $\quad \lim\limits_{n\to\infty} (\lambda z_n) = \lambda\alpha, \quad$ where $\lambda \in \mathbb{C}$;

Product Rule $\quad \lim\limits_{n\to\infty} (z_n w_n) = \alpha\beta$;

Quotient Rule $\quad \lim\limits_{n\to\infty} \left(\dfrac{z_n}{w_n}\right) = \dfrac{\alpha}{\beta},$
provided that $\beta \neq 0$.

7. If $\lim\limits_{n\to\infty} z_n = \alpha$, then

(a) $\lim\limits_{n\to\infty} |z_n| = |\alpha|$;

(b) $\lim\limits_{n\to\infty} \overline{z_n} = \overline{\alpha}$;

(c) $\lim\limits_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} \alpha$;

(d) $\lim\limits_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} \alpha$.

8. A sequence which is not convergent is **divergent.**

9. The sequence $\{z_n\}$ **tends to infinity** if, for each positive number M, there is an integer N such that

$|z_n| > M, \quad$ for all $n > N$.

In this case we write

$z_n \to \infty$ as $n \to \infty$.

10. Reciprocal Rule

Let $\{z_n\}$ be a sequence. Then

$z_n \to \infty$ as $n \to \infty$

if and only if

$\{1/z_n\}$ is a null sequence.

11. Let $\{n_k\}$ be a sequence of positive integers which are strictly increasing; that is,

$n_1 < n_2 < n_3 < \ldots.$

Then the sequence $\{z_{n_k}\}$ is a **subsequence** of the sequence $\{z_n\}$.

In particular, $\{z_{2k}\}$ is the **even subsequence** and $\{z_{2k-1}\}$ is the **odd subsequence.**

12. Subsequence Rules

(a) **First Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has a subsequence which tends to infinity.

13. (a) If $|\alpha| > 1$, then the sequence $\{\alpha^n\}$ tends to infinity.

(b) If $|\alpha| = 1$ and $\alpha \neq 1$, then the sequence $\{\alpha^n\}$ is divergent.

14. If $\{z_n\}$ is a convergent sequence, then there is a positive number M such that

$|z_n| \le M, \quad$ for $n = 1, 2, \ldots.$

In this case, we say that $\{z_n\}$ is **bounded.**

Section 2: Continuous Functions

1. Continuity: sequential definition

Let $f : A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each sequence $\{z_n\}$ in A such that $z_n \to \alpha$, we have

$$f(z_n) \to f(\alpha);$$

that is,

$$z_n \to \alpha \implies f(z_n) \to f(\alpha).$$

If f is continuous at each α in A, then we say that f is **continuous** (on A).

If f is not continuous at α, then we say that f is **discontinuous** at α.

2. Continuity: ε-δ definition

Let $f : A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$z \in A, \ |z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$

3. The ε-δ definition of continuity is equivalent to the sequential definition of continuity.

4. Basic continuous functions

(a) polynomial and rational functions;

(b) $f(z) = |z|, \overline{z}, \operatorname{Re} z, \operatorname{Im} z$;

(c) $f(z) = e^z$;

(d) trigonometric and hyperbolic functions;

(e) $f(z) = \operatorname{Arg} z, \operatorname{Log} z, z^\alpha$, on $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$.

5. Combination Rules

If the functions f and g are continuous at α, then so are

Sum Rule $f + g$;
Multiple Rule λf, for $\lambda \in \mathbb{C}$;
Product Rule fg;
Quotient Rule f/g, provided that $g(\alpha) \neq 0$.

6. Composition Rule

If the function f is continuous at α, and if the function g is continuous at $f(\alpha)$, then $g \circ f$ is continuous at α.

7. Restriction Rule

If the function f has domain A, the function g has domain B and

1. f is the restriction of g to A;
2. g is continuous at $\alpha \in A$,

then f is continuous at α.

Section 3: Limits of Functions

1. The point α is a **limit point** of a set A if there is a sequence $\{z_n\}$ such that

$$z_n \in A - \{\alpha\}, \quad \text{for } n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} z_n = \alpha.$$

2. Let f be a function with domain A, and suppose that α is a limit point of A. Then the function f has **limit β as z tends to α** if for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$$f(z_n) \to \beta.$$

In this case we write

EITHER $\lim\limits_{z\to\alpha} f(z) = \beta$,

OR $f(z) \to \beta$ as $z \to \alpha$.

3. Strategy for proving that a limit does not exist

To prove that $\lim\limits_{z\to\alpha} f(z)$ does not exist, where α is a limit point of the domain A of the function f:

EITHER

(a) find two sequences $\{z_n\}$ and $\{z_n'\}$ in $A - \{\alpha\}$ which both tend to α, such that the sequences $\{f(z_n)\}$ and $\{f(z_n')\}$ have different limits;

OR

(b) find a sequence $\{z_n\}$ in $A - \{\alpha\}$ which tends to α, such that the sequence $\{f(z_n)\}$ tends to infinity.

4. Let f be a function with domain A and suppose that $\alpha \in A$ is a limit point of A. Then

$$f \text{ is continuous at } \alpha \iff \lim_{z\to\alpha} f(z) = f(\alpha).$$

5. Let f be a function with domain A and suppose that α is a limit point of A. Then the function f has **limit β as z tends to α** if for each positive ε, there is a positive δ such that

$$|f(z) - \beta| < \varepsilon, \quad \text{for all } z \in A - \{\alpha\} \text{ with } |z - \alpha| < \delta.$$

6. Combination Rules

Let f and g be functions with domains A and B, respectively, and suppose that α is a limit point of $A \cap B$. If

$$\lim_{z\to\alpha} f(z) = \beta \quad \text{and} \quad \lim_{z\to\alpha} g(z) = \gamma,$$

then

Sum Rule $\lim\limits_{z\to\alpha} (f(z) + g(z)) = \beta + \gamma$;
Multiple Rule $\lim\limits_{z\to\alpha} (\lambda f(z)) = \lambda\beta$, for $\lambda \in \mathbb{C}$;
Product Rule $\lim\limits_{z\to\alpha} (f(z)g(z)) = \beta\gamma$;
Quotient Rule $\lim\limits_{z\to\alpha} (f(z)/g(z)) = \beta/\gamma$, provided that $\gamma \neq 0$.

Section 4: Regions

1. A set A in $\mathbb{C}$ is **open** if each point α in A is the centre of some open disc lying entirely in A.

2. **Combination Rules**

If A_1 and A_2 are open sets, then so are:

(a) $A_1 \cup A_2$ and

(b) $A_1 \cap A_2$.

This rule extends to n open sets $A_1, A_2, \ldots, A_n$:

$$A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{and} \quad A_1 \cap A_2 \cap \ldots \cap A_n$$

are open sets.

3. A set A in $\mathbb{C}$ is **(pathwise) connected** if each pair of points α, β in A can be joined by a path lying entirely in A.

4. Connected sets in which any two points α and β can be joined by a line segment are called **convex**.

5. Let f be a continuous function whose domain A is connected. Then the image $f(A)$ is also connected.

6. A **region** is a non-empty, connected, open set.

7. **Basic regions**

(a) any open disc;

(b) any open half-plane;

(c) the complement of any closed disc;

(d) any open annulus;

(e) any open rectangle;

(f) any open sector (including cut planes);

(g) the set $\mathbb{C}$ itself.

8. If $\mathcal{R}$ is a region and $\alpha_0 \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_0\}$ is also a region.

Section 5: The Extreme Value Theorem

1. A set E in $\mathbb{C}$ is **closed** if its complement $\mathbb{C} - E$ is open.

2. If E is a closed set and $\{z_n\}$ is a sequence in E with limit α, then $\alpha \in E$.

3. **Combination Rules**

If E_1 and E_2 are closed sets, then so are:

(a) $E_1 \cup E_2$ and

(b) $E_1 \cap E_2$.

This rule extends to n closed sets $E_1, E_2, \ldots, E_n$:

$$E_1 \cup E_2 \cup \ldots \cup E_n \quad \text{and} \quad E_1 \cap E_2 \cap \ldots \cap E_n$$

are closed sets.

4. A set E in $\mathbb{C}$ is **bounded** if it is contained in some closed disc.

A set is **unbounded** if it is not bounded.

5. A set E in $\mathbb{C}$ is **compact** if E is closed and bounded.

6. **Extreme Value Theorem**

Let the function f be continuous on a compact set E. Then there are numbers α, β in E such that

$$|f(\beta)| \leq |f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

7. **Boundedness Theorem**

Let the function f be continuous on a compact set E. Then there is a number M such that

$$|f(z)| \leq M, \quad \text{for all } z \in E.$$

8. If the function f is continuous on a compact set E, then $f(E)$ is compact.

9. Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then

(a) α is an **interior point** of A if there is an open disc centred at α which lies entirely in A;

(b) α is an **exterior point** of A if there is an open disc centred at α which lies entirely outside A.

The set of interior points of A forms the **interior** of A, written $\operatorname{int} A$, and the set of exterior points of A forms the **exterior** of A, written $\operatorname{ext} A$.

10. Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

The set of boundary points of A forms the **boundary** of A, written ∂A.

The sets $\operatorname{int} A$, $\operatorname{ext} A$ and ∂A are disjoint and

$$\partial A = \mathbb{C} - (\operatorname{int} A \cup \operatorname{ext} A).$$

Also $\operatorname{int} A$ and $\operatorname{ext} A$ are open, whereas ∂A is closed.

11. **Nested Rectangles Theorem**

Let $R_n, n = 0, 1, 2, \ldots$, be a sequence of closed rectangles with sides parallel to the axes, and with diagonals of lengths $s_n, n = 0, 1, 2, \ldots$, such that

1. $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots$, and
2. $\lim_{n \to \infty} s_n = 0$.

Then there is a unique complex number α which lies in all of the rectangles R_n. Moreover, for each positive number ε, there is an integer N such that

$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N.$$

Unit A4 Differentiation

Section 1: Derivatives of Complex Functions

1. Let f be a complex function whose domain contains the point α. Then the **derivative of f at α** is

$$\lim_{z\to\alpha}\frac{f(z)-f(\alpha)}{z-\alpha} \quad \left(\text{or} \quad \lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h}\right),$$

provided that this limit exists. If it does exist, then f is **differentiable at α**. If f is differentiable at every point of a set A, then f is **differentiable on A**. A function is **differentiable** if it is differentiable on its domain.

The derivative of f at α is denoted by $f'(\alpha)$, and the function

$$f' : z \longmapsto f'(z)$$

is called the **derivative** of f. The domain of f' is the set of all complex numbers at which f is differentiable.

2. A function is **entire** if it is differentiable on the whole of $\mathbb{C}$.

3. If a function f is differentiable on a region $\mathcal{R}$, then f is said to be **analytic on $\mathcal{R}$**. If the domain of f is a region and if f is differentiable on its domain, then f is said to be **analytic**. A function f is **analytic at a point α** if it is differentiable on a region containing α.

4. If the complex function f is differentiable at α, then f is continuous at α.

5. **Linear Approximation Theorem**

If the complex function f is differentiable at α, then f may be approximated near α by a linear polynomial. More precisely,

$$f(z) = f(\alpha) + (z-\alpha)f'(\alpha) + e(z),$$

where e is an 'error' function satisfying $e(z)/(z-\alpha) \to 0$ as $z \to \alpha$.

6. **Combination Rules**

Let f and g be complex functions with domains A and B, respectively, and let α be a limit point of $A \cap B$. If f and g are differentiable at α, then

Sum Rule $\quad (f+g)'(\alpha) = f'(\alpha) + g'(\alpha)$;

Multiple Rule $\quad (\lambda f)'(\alpha) = \lambda f'(\alpha)$;

Product Rule $\quad (fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha)$;

Quotient Rule $\quad (f/g)'(\alpha) = \dfrac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}$,

provided that $g(\alpha) \neq 0$.

7. A polynomial function

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (z \in \mathbb{C}),$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$, is entire with derivative

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} \quad (z \in \mathbb{C}).$$

Any rational function is analytic.

8. **Strategy A for non-differentiability**

If f is discontinuous at α, then f is not differentiable at α.

9. **Strategy B for non-differentiability**

To prove that a function f is not differentiable at α, apply the strategy for the non-existence of limits to the difference quotient

$$\frac{f(z)-f(\alpha)}{z-\alpha}.$$

10. **Higher-order derivatives** of a function f are obtained by repeated differentiation:

$$(f')' = f'' = f^{(2)},$$
$$(f'')' = f''' = f^{(3)},$$

and so on.

The **nth derivative** of f is denoted by $f^{(n)}$.

11. **A geometric interpretation of derivatives**

If $f'(\alpha) \neq 0$, then, to a close approximation,

$$f(z) - f(\alpha) \cong f'(\alpha)(z-\alpha).$$

It follows that, to a close approximation, a small disc centred at α is mapped to a small disc centred at $f(\alpha)$. In this process, the disc is rotated through the angle $\operatorname{Arg} f'(\alpha)$, and scaled by the factor $|f'(\alpha)|$.

Section 2: The Cauchy–Riemann Equations

1. **Cauchy–Riemann Theorem**

If f is differentiable at $\alpha = a + ib$, then $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist at (a, b) and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b).$$

2. **Strategy C for non-differentiability**

Let $f(x+iy) = u(x,y) + iv(x,y)$. If either

$$\frac{\partial u}{\partial x}(a,b) \neq \frac{\partial v}{\partial y}(a,b) \quad \text{or} \quad \frac{\partial v}{\partial x}(a,b) \neq -\frac{\partial u}{\partial y}(a,b),$$

then f is not differentiable at $a + ib$.

3. **Cauchy–Riemann Converse Theorem**

Let $f(x+iy) = u(x,y) + iv(x,y)$ be defined on a region $\mathcal{R}$ containing $a + ib$. If the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$

1. exist on $\mathcal{R}$,
2. are continuous at (a, b),
3. satisfy the Cauchy–Riemann equations at (a, b),

then f is differentiable at $a + ib$ and

$$f'(a+ib) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b).$$

Section 3: The Composition, Inverse and Restriction Rules

1. **Composition Rule**

Let f and g be complex functions, and let α be a limit point of the domain of $g \circ f$. If f is differentiable at α, and g is differentiable at $f(\alpha)$, then $g \circ f$ is differentiable at α, and

$$(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).$$

2. **Inverse Function Rule**

Let $f : A \longrightarrow B$ be a one-one complex function, and suppose that f^{-1} is continuous at $\beta \in B$. If f has a non-zero derivative at $f^{-1}(\beta) \in A$, then f^{-1} is differentiable at β and $\left(f^{-1}\right)'(\beta) = \frac{1}{f'\left(f^{-1}(\beta)\right)}$.

3. **Restriction Rule**

Let f and g be functions with domains A and B, respectively, and let $A \subseteq B$. If $\alpha \in A$ is a limit point of A and

1. $f(z) = g(z)$, for $z \in A$,
2. g is differentiable at α,

then f is differentiable at α, and $f'(\alpha) = g'(\alpha)$.

4. **Standard derivatives**

Rule of f	Rule of f'	Domain of f'
$\alpha,\ \alpha \in \mathbb{C}$	0	$\mathbb{C}$
$z^k,\ k \in \mathbb{Z},\ k > 0$	kz^{k-1}	$\mathbb{C}$
$z^k,\ k \in \mathbb{Z},\ k < 0$	kz^{k-1}	$\mathbb{C} - \{0\}$
$z^\alpha,\ \alpha \in \mathbb{C} - \mathbb{Z}$	$\alpha z^{\alpha-1}$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\exp z$	$\exp z$	$\mathbb{C}$
$\operatorname{Log} z$	$1/z$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\sin z$	$\cos z$	$\mathbb{C}$
$\cos z$	$-\sin z$	$\mathbb{C}$
$\tan z$	$\sec^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$
$\sinh z$	$\cosh z$	$\mathbb{C}$
$\cosh z$	$\sinh z$	$\mathbb{C}$
$\tanh z$	$\operatorname{sech}^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi i : n \in \mathbb{Z}\right\}$

Section 4: Smooth Paths

1. Let Γ be a path with parametrization $\gamma : I \longrightarrow \mathbb{C}$, and suppose that $c \in I$. If γ is differentiable at c and if $\gamma'(c) \neq 0$, then $\gamma'(c)$ may be interpreted as a tangent vector to the path Γ at the point $\gamma(c)$.

2. Let ϕ and ψ be real functions, both with domain some interval I. Then the parametrization

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$

is differentiable at a point $c \in I$ if and only if both ϕ and ψ are differentiable at c. If ϕ and ψ are differentiable at c, then

$$\gamma'(c) = \phi'(c) + i\psi'(c).$$

3. A parametrization $\gamma : I \longrightarrow \mathbb{C}$ is **smooth** if

(a) γ is differentiable on I;

(b) γ' is continuous on I;

(c) γ' is non-zero on I.

A path is **smooth** if its parametrization is smooth.

4. Let f be a function that is analytic on a region $\mathcal{R}$, and suppose that $f'(\alpha) \neq 0$ for some $\alpha \in \mathcal{R}$. If $\Gamma : \gamma$ is a smooth path in $\mathcal{R}$, passing through α, then the tangent vector to $f(\Gamma)$ at $f(\alpha)$ may be obtained from the tangent vector to Γ at α by a rotation through the angle $\operatorname{Arg} f'(\alpha)$ and a scaling by the factor $|f'(\alpha)|$.

5. A function is **conformal at α** if it leaves unchanged the size and orientation of the angle between any two smooth paths through α. A function is **conformal on a set S** if it is conformal at every point of S. A function is **conformal** if it is conformal on its domain.

6. Let f be a function which is analytic at α, with $f'(\alpha) \neq 0$. Then f is conformal at α.

7. Smooth paths that meet at right angles are said to be **orthogonal**. A grid made up of orthogonal smooth paths is called an **orthogonal grid**.

PART III: INDEX

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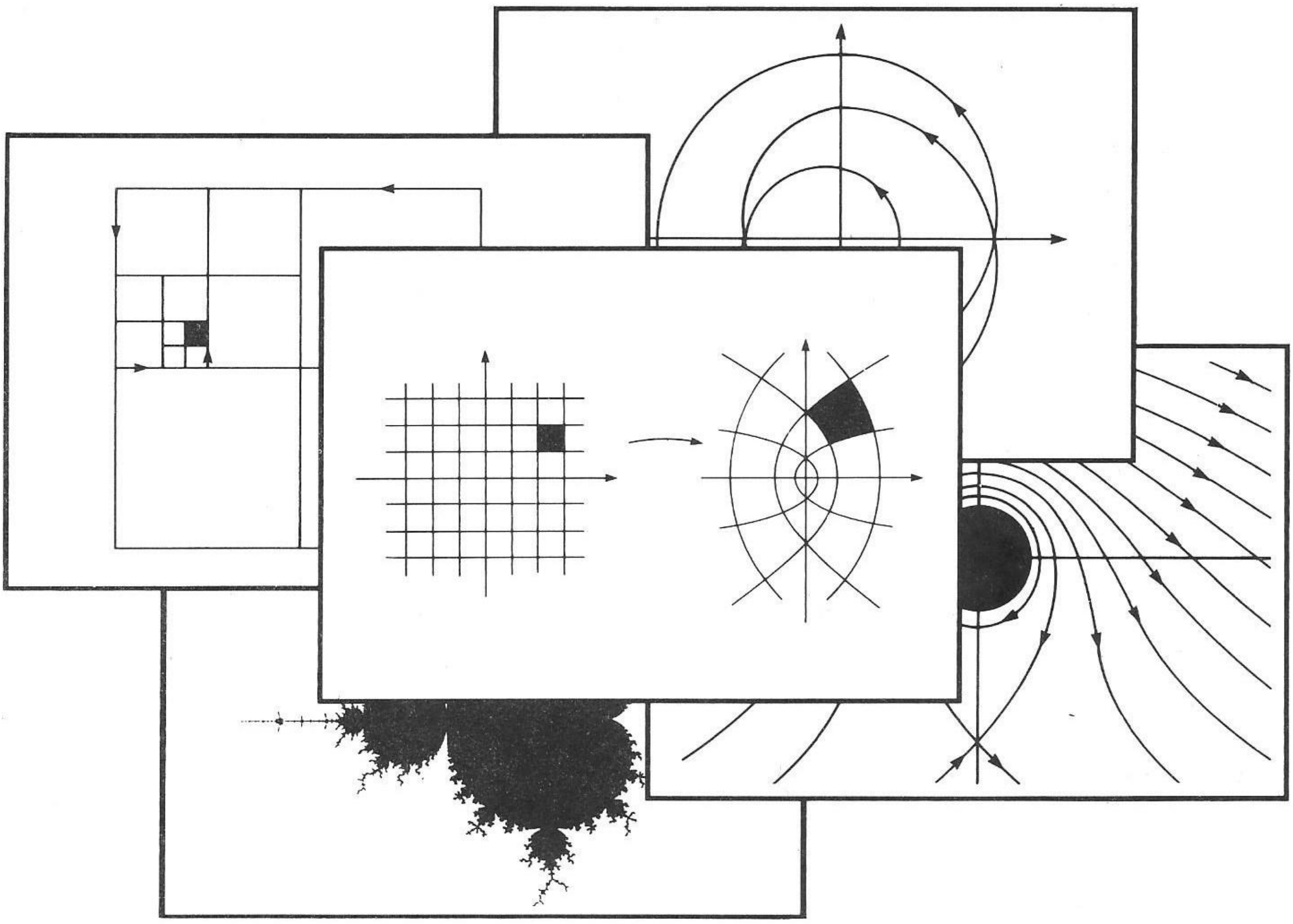
COMPLEX ANALYSIS

HANDBOOK TWO



COMPLEX ANALYSIS COMPLEX ANALYSIS COMPL

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CONTENTS

PART I: UNIT SUMMARIES

Unit B1 Integration

Section 1: Integrating Real Functions

This section revises the definition and main properties of the integration of real functions.

Section 2: Integrating Complex Functions

1. Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a smooth path in $\mathbb{C}$, and let f be a function which is continuous on Γ. Then the **integral of f along the path Γ**, denoted by $\int_\Gamma f(z)\,dz$ or $\int_\Gamma f$, is

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

This integral is evaluated as follows:

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

where $u(t) = \operatorname{Re}(f(\gamma(t))\gamma'(t))$ and $v(t) = \operatorname{Im}(f(\gamma(t))\gamma'(t))$.

2. The above definition may be remembered by writing

$$z = \gamma(t), \quad dz = \gamma'(t)\,dt.$$

3. Two parametrizations

$$\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C} \quad \text{and} \quad \gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$$

are **equivalent** if there is a function $h : [a_1, b_1] \longrightarrow [a_2, b_2]$ satisfying

(a) $h(a_1) = a_2, h(b_1) = b_2$,

(b) h' exists on $[a_1, b_1]$, and is continuous and positive there,

such that

$$\gamma_1(t) = \gamma_2(h(t)), \quad \text{for } t \in [a_1, b_1].$$

If γ_1 and γ_2 are equivalent parametrizations, then the paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are **equivalent paths**.

4. If the smooth paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are equivalent and the function f is continuous on Γ, then $\int_\Gamma f(z)\,dz$ does not depend on which parametrization, γ_1 or γ_2, is used.

5. A **contour** Γ is a path which can be subdivided into a finite number of smooth paths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, joined end to end; the order of these constituent smooth paths is indicated by writing

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n.$$

The **initial point** of Γ is the initial point of Γ_1, and the **final point** of Γ is the final point of Γ_n.

6. Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ be a contour, and let f be a function which is continuous on Γ. Then the **(contour) integral of f along Γ**, denoted by $\int_\Gamma f(z)\,dz$ or $\int_\Gamma f$, is

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz.$$

7. All splittings of a contour into smooth paths give the same value for the contour integral.

8. **Combination Rules**
Let Γ be a contour, and let f and g be functions which are continuous on Γ. Then

Sum Rule

$$\int_\Gamma (f(z) + g(z))\,dz = \int_\Gamma f(z)\,dz + \int_\Gamma g(z)\,dz;$$

Multiple Rule

$$\int_\Gamma \lambda f(z)\,dz = \lambda \int_\Gamma f(z)\,dz, \quad \text{for } \lambda \in \mathbb{C}.$$

9. Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a smooth path. Then the **reverse path**, denoted by $\widetilde{\Gamma}$, is the path with parametrization $\widetilde{\gamma}(t)$, where

$$\widetilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a,b]).$$

As sets, Γ and $\widetilde{\Gamma}$ are the same.

10. If $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ is a contour, then the **reverse contour** $\widetilde{\Gamma}$ of Γ is

$$\widetilde{\Gamma} = \widetilde{\Gamma}_n + \widetilde{\Gamma}_{n-1} + \cdots + \widetilde{\Gamma}_1.$$

11. **Reverse Contour Theorem**
Let Γ be a contour, and let f be a function which is continuous on Γ. Then, if $\widetilde{\Gamma}$ is the reverse contour of Γ,

$$\int_{\widetilde{\Gamma}} f(z)\,dz = -\int_\Gamma f(z)\,dz.$$

Section 3: Evaluating Contour Integrals

1. Let f and F be functions defined on a region $\mathcal{R}$. Then F is a **primitive of f on $\mathcal{R}$** if F is analytic on $\mathcal{R}$ and

$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$

The function F is also called an **antiderivative** or **indefinite integral** of f on $\mathcal{R}$.

2. **Fundamental Theorem of Calculus**
Let the function f be continuous on $\mathcal{R}$, let F be a primitive of f on $\mathcal{R}$, and let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then

$$\int_\Gamma f(z)\,dz = F(\beta) - F(\alpha) = [F(z)]_\alpha^\beta.$$

3. The Chain Rule,

$$g'(f(z))f'(z) = (g \circ f)'(z),$$

provides a primitive $g \circ f$ for the integrand in

$$\int_\Gamma g'(f(z))f'(z)\,dz.$$

This is a form of integration by substitution.

4. **Contour Independence Theorem**

Let the function f be continuous and have a primitive on a region $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

5. **Integration by Parts**

Let the functions f and g be analytic on a region $\mathcal{R}$, and let f' and g' be continuous on $\mathcal{R}$. Let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then

$$\int_\Gamma f(z)g'(z)\,dz = [f(z)g(z)]_\alpha^\beta - \int_\Gamma f'(z)g(z)\,dz.$$

6. The Fundamental Theorem of Calculus cannot be used to integrate non-differentiable functions.

7. A path or contour Γ is **closed** if its initial and final points coincide.

If Γ is a closed contour, then the value of any contour integral along Γ does not depend on the choice of initial (= final) point.

8. **Closed Contour Theorem**

Let the function f be continuous and have a primitive F on a region $\mathcal{R}$. Then

$$\int_\Gamma f(z)\,dz = 0,$$

for any closed contour Γ in $\mathcal{R}$.

9. A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.

10. **Grid Path Theorem**

If $\mathcal{R}$ is a region, then any two points in $\mathcal{R}$ can be joined by a grid path in $\mathcal{R}$.

11. **Zero Derivative Theorem**

Let the function F be analytic on a region $\mathcal{R}$, and let $F'(z) = 0$, for all z in $\mathcal{R}$. Then F is constant on $\mathcal{R}$.

Section 4: Estimating Contour Integrals

1. Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a smooth path. Then the **length of the path Γ**, denoted by $L(\Gamma)$, is

$$L(\Gamma) = \int_a^b |\gamma'(t)|\,dt.$$

The **length of a contour** is the sum of the lengths of its constituent smooth paths.

2. Equivalent smooth paths have the same length. A smooth path and its reverse path have the same length.

3. **Estimation Theorem**

Let f be a function which is continuous on a contour Γ of length L, with

$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$

Then

$$\left|\int_\Gamma f(z)\,dz\right| \le ML.$$

4. Let $g : [a, b] \longrightarrow \mathbb{C}$ be a continuous function. Then

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt.$$

Unit B2 Cauchy's Theorem

Section 1: Cauchy's Theorem

1. A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is **simple-closed** if it is closed and γ is one-one on $[a, b[$.

A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is **simple** if γ is one-one on $[a, b]$.

2. Jordan Curve Theorem

If Γ is a simple-closed path, then the complement $\mathbb{C} - \Gamma$ of Γ is the union of two disjoint regions:

a bounded region, called the **inside** of Γ, and

an unbounded region, called the **outside** of Γ.

3. A region $\mathcal{R}$ is **simply-connected** if, whenever Γ is a simple-closed path lying in $\mathcal{R}$, the inside of Γ also lies in $\mathcal{R}$.

In practice, we usually employ the following *informal definition* when testing a region for simple-connectedness:

a region is simply-connected if it has no holes in it.

4. Cauchy's Theorem

Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then

$$\int_\Gamma f(z)\,dz = 0,$$

for any closed contour Γ in $\mathcal{R}$.

5. Contour Independence Theorem

Let $\mathcal{R}$ be a simply-connected region, let f be a function which is analytic on $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

6. Convention Unless otherwise specified, any simple-closed contour in a contour integral will be assumed to be traversed once anticlockwise.

7. Shrinking Contour Theorem

Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, let α be a point inside Γ, and let g be a function which is analytic on $\mathcal{R} - \{\alpha\}$. Then

$$\int_\Gamma g(z)\,dz = \int_C g(z)\,dz,$$

where C is any circle with centre α, lying inside Γ.

Section 2: The Integral Formula

1. Cauchy's Integral Formula

Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then

$$f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\alpha}\,dz,$$

for any point α inside Γ.

2. Liouville's Theorem

If f is a bounded entire function, then f is constant.

Section 3: The Derivative Formulas

1. Cauchy's nth Derivative Formula

Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then, for any point α inside Γ, f is n-times differentiable at α and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz, \quad \text{for } n = 1, 2, \ldots.$$

2. Analyticity of Derivatives

Let $\mathcal{R}$ be a region, and let f be a function which is analytic on $\mathcal{R}$. Then f possesses derivatives of all orders on $\mathcal{R}$, so that $f', f'', f^{(3)}, \ldots$ are analytic on $\mathcal{R}$.

Section 4: Revision

1. Contour integrals may be evaluated by the following methods:

parametrization (using the definition of a contour integral) — *Unit B1*,

Closed Contour Theorem — *Unit B1*,

Cauchy's Theorem — Section 1,

Cauchy's Integral Formula — Section 2,

Cauchy's nth Derivative Formula — Section 3.

2. If the value of a contour integral is known, then evaluating the integral using parametrization yields the value of two real integrals.

3. **Strategy** To evaluate $I = \int_\Gamma \frac{g(z)}{p(z)}\,dz$, where g is analytic on a simply-connected region containing a simple-closed contour Γ, and p is a polynomial function with no zeros on Γ:

   (a) factorize $p(z)$ as $q(z)r(z)$, where the zeros of q lie inside Γ and those of r lie outside Γ; then $f = g/r$ is analytic on $\mathcal{R}$, a simply-connected region which contains Γ;

   (b) expand $1/q(z)$ in partial fractions;

   (c) hence expand $I = \int_\Gamma \frac{f(z)}{q(z)}\,dz$ as a sum of integrals, each of which can be evaluated using Cauchy's Integral Formula and/or Cauchy's nth Derivative Formula.

Section 5: The proof of Cauchy's Theorem

1. **Cauchy's Theorem for a rectangular contour**
   Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
   $$\int_\Gamma f(z)\,dz = 0,$$
   for any rectangular contour Γ in $\mathcal{R}$.

   (This theorem is used in the proof of the following theorem.)

2. **Cauchy's Theorem for a closed grid path**
   Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
   $$\int_\Gamma f(z)\,dz = 0,$$
   for any closed grid path Γ in $\mathcal{R}$.

   (This theorem is used in the proof of the following theorem.)

3. **Primitive Theorem** (or **Antiderivative Theorem**)
   If a function f is analytic on a simply-connected region $\mathcal{R}$, then f has a primitive on $\mathcal{R}$.

4. Cauchy's Theorem (see Section 1) is proved by using the Primitive Theorem and then appealing to the Closed Contour Theorem (*Unit B1*).

5. **Morera's Theorem**
   If a function f is continuous on a region $\mathcal{R}$ and $\int_\Gamma f(z)\,dz = 0$, for all rectangular contours Γ in $\mathcal{R}$, then f is analytic on $\mathcal{R}$.

Unit B3 Taylor Series

Section 1: Complex Series

1. The expression $z_1 + z_2 + z_3 + \cdots$ is called an **infinite series** or a **series**. The number z_n is called the **nth term** of the series. The **nth partial sum** of the series is
   $$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k.$$

2. The series $z_1 + z_2 + z_3 + \cdots$ is **convergent with sum** s if the sequence $\{s_n\}$ of partial sums converges to s. We say that the series **converges to** s, and write
   $$z_1 + z_2 + z_3 + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$
   The series **diverges** if the sequence $\{s_n\}$ diverges.

3. If $\sum_{n=1}^{\infty} z_n$ is a convergent series, then $\{z_n\}$ is a null sequence.

4. **Non-null Test**
   If the sequence $\{z_n\}$ is not null, then the series $\sum_{n=1}^{\infty} z_n$ is divergent.

5. The series $\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots$, where $a \in \mathbb{C}$, is called a **geometric series with common ratio** z.

6. **Geometric series**

   (a) If $|z| < 1$ and $a \in \mathbb{C}$, then the series $\sum_{n=0}^{\infty} az^n$ is convergent with sum $a/(1-z)$.

   (b) If $|z| \geq 1$ and $a \in \mathbb{C} - \{0\}$, then the series $\sum_{n=0}^{\infty} az^n$ is divergent.

7. The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called the **harmonic series**.

8. The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$, and diverges if $p \leq 1$.

9. **Combination Rules**
   If $\sum_{n=1}^{\infty} z_n = s$ and $\sum_{n=1}^{\infty} w_n = t$, then

   Sum Rule $\quad \sum_{n=1}^{\infty}(z_n + w_n) = s + t;$

   Multiple Rule $\quad \sum_{n=1}^{\infty} \lambda z_n = \lambda s, \quad$ for $\lambda \in \mathbb{C}$.

10. The series $\sum_{n=1}^{\infty} z_n$ is convergent if and only if both the series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ are convergent. In this case

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i\sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

11. Comparison Test

If $\sum_{n=1}^{\infty} a_n$ is a real convergent series of positive terms, and if

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \ldots,$$

then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

12. The series $\sum_{n=1}^{\infty} z_n$ is **absolutely convergent** if the real series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

13. Absolute Convergence Test

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

14. Triangle Inequality

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|.$$

15. Ratio Test

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series for which

$$\left|\frac{z_{n+1}}{z_n}\right| \to l \text{ as } n \to \infty.$$

(a) If $0 \leq l < 1$, then $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

(b) If $l > 1$, then $\sum_{n=1}^{\infty} z_n$ is divergent.

The case $l > 1$ includes the situation where $\left|\frac{z_{n+1}}{z_n}\right| \to \infty$ as $n \to \infty$.

Section 2: Power Series

1. Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

where $\alpha \in \mathbb{C}, a_n \in \mathbb{C}, n = 0, 1, 2, \ldots$, is called a **power series about α.**

2. A power series **converges on a set S** if for each $z \in S$, the corresponding series converges.

3. Let $A = \{z : \sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ converges}\}$. The function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad (z \in A)$$

is called the **sum function** of the power series.

4. Radius of Convergence Theorem

For a given power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

precisely one of the following possibilities occurs:

(a) the series converges only for $z = \alpha$;

(b) the series converges for all z;

(c) there is a number $R > 0$ such that

$\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ converges (absolutely) if $|z-\alpha| < R$,

and

$\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ diverges if $|z-\alpha| > R$.

5. The positive real number R in case (c) of the Radius of Convergence Theorem is called the **radius of convergence** of the power series. Case (a) corresponds to writing $R = 0$, and case (b) to writing $R = \infty$.

6. All the convergence tests of Section 1 can be applied to power series. In particular, the Ratio Test can be used to find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ provided that

$$\frac{|a_{n+1}(z-\alpha)^{n+1}|}{|a_n(z-\alpha)^n|}$$

tends to a limit (possibly ∞) as $n \to \infty$. In this case, the radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

7. Let R be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n.$$

The **disc of convergence** of the power series is the open disc $\{z : |z-\alpha| < R\}$. The disc of convergence is interpreted to be $\varnothing$ if $R = 0$, and to be $\mathbb{C}$ if $R = \infty$.

8. A power series may converge at none, some or all of the points on the boundary of its disc of convergence.

9. **Differentiation Rule**
The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1}$$

have the same radius of convergence R, say.
Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$, then f is analytic on the disc of convergence $\{z : |z-\alpha| < R\}$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1}, \quad \text{for } |z-\alpha| < R.$$

10. **Integration Rule**
The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

have the same radius of convergence R, say.
Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$, then the function

$$F(z) = \text{constant} + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

is a primitive of f on $\{z : |z-\alpha| < R\}$.

Section 3: Taylor's Theorem

1. **Taylor's Theorem**
If f is a function which is analytic on the open disc $D = \{z : |z-\alpha| < r\}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n, \quad \text{for } z \in D. \qquad (*)$$

Moreover, this representation of f is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } z \in D,$$

then $a_n = f^{(n)}(\alpha)/n!$, for $n = 0, 1, 2, \ldots$.
(In $(*)$, we have employed the conventions $0! = 1, 0^0 = 1$ and $f^{(0)}(z) = f(z)$.)

2. If f is a function with derivatives $f'(\alpha), f''(\alpha), f'''(\alpha), \ldots$ at the point α, then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

is called the **Taylor series about α for f**. The coefficient $f^{(n)}(\alpha)/n!$ is known as the **nth Taylor coefficient** (of f at α).

3. If f is an entire function, then the Taylor series about any point α for f converges to $f(z)$ for all $z \in \mathbb{C}$.

4. A function $f : A \longrightarrow \mathbb{C}$ is **even** if

$$f(-z) = f(z), \quad \text{for } z \in A,$$

and **odd** if

$$f(-z) = -f(z), \quad \text{for } z \in A.$$

The Taylor series about 0 of an even (odd) function includes only even (odd) powers of z.

5. **Basic Taylor Series**

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1;$$

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1;$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

6. **Binomial series**
If $\alpha \in \mathbb{C}$, then the **binomial series** about 0 for the function $f(z) = (1+z)^\alpha$ is

$$(1+z)^\alpha = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \binom{\alpha}{3} z^3 + \cdots, \quad \text{for } |z| < 1,$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}$.

Section 4: Manipulating Taylor Series

1. The restrictions of the functions tan and sin to the region $S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ have inverse functions denoted by $\tan^{-1}$ and $\sin^{-1}$. The domains of both of these inverse functions contain the open unit disc with centre zero, and their derivatives have the following rules:

$$(\tan^{-1})'(z) = \frac{1}{1+z^2}; \quad (\sin^{-1})'(z) = \frac{1}{\sqrt{1-z^2}}.$$

2. **Product Rule**
Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R'.$$

If $r = \min\{R, R'\}$, then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < r,$$

where, for each m, the numbers $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\left(\sum_{k=0}^{m} a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m} b_k(z-\alpha)^k\right).$$

3. **Composition Rule**
Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(w) = \sum_{n=0}^{\infty} b_n(w-\beta)^n, \quad \text{for } |w-\beta| < R'.$$

If $\beta = f(\alpha)$ $(= a_0)$, then, for some $r > 0$,

$$g(f(z)) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < r,$$

where, for each m, the numbers $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\sum_{k=0}^{m} b_k \left(\sum_{l=1}^{m} a_l(z-\alpha)^l\right)^k.$$

4. Taylor series may be deduced from known ones (for example, the basic Taylor series about 0 listed in Section 3) by:
   (a) using the Combination Rules (Section 1);
   (b) using the Product Rule;
   (c) using the Composition Rule;
   (d) using the Differentiation Rule (Section 2);
   (e) using the Integration Rule (Section 2);
   (f) using the substitution $w = \lambda z^k$ (where $\lambda \neq 0$ and $k \in \{1, 2, 3, \ldots\}$), for which
   $$|w| < R \iff |z| < \sqrt[k]{R/|\lambda|};$$
   (g) using the substitution $w = z + (\beta - \alpha)$, which changes a Taylor series about β to one about α and for which
   $$|w-\beta| < R \iff |z-\alpha| < R;$$
   (h) using appropriate combinations of methods (a)–(g).

Section 5: The Uniqueness Theorem

1. Let a function f be analytic at α. If
$$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0,$$
but $f^{(k)}(\alpha) \neq 0$,
then f has a **zero at α of (finite) order k**.
A **simple zero** is a zero of order 1.

2. A function f is analytic at a point α, and has a zero of order k at α, if and only if, for some $r > 0$,
$$f(z) = (z-\alpha)^k g(z), \quad \text{for } |z-\alpha| < r,$$
where g is analytic at α, and $g(\alpha) \neq 0$.

3. If a function f is analytic on a region $\mathcal{R}$ and not identically zero there, then any zero of f is of finite order.

4. A zero α of a function f is said to be **isolated** if some disc centred at α contains no other zeros of f.

5. **Isolated zeros**
A zero of finite order is isolated.

6. If a function f is analytic on a region $\mathcal{R}$ and if S is a set of zeros of f with a limit point in $\mathcal{R}$, then f is identically zero on $\mathcal{R}$ (that is, $f(z) = 0$, for all $z \in \mathcal{R}$).

7. **Uniqueness Theorem**
Let the functions f and g be analytic on a region $\mathcal{R}$ and suppose that f agrees with g throughout a set $S \subseteq \mathcal{R}$, where S has a limit point in $\mathcal{R}$. Then f agrees with g throughout $\mathcal{R}$.

Unit B4 Laurent Series

Section 1: Singularities

1. A function f has an (isolated) **singularity** at the point α if f is analytic on a punctured open disc $\{z : 0 < |z - \alpha| < r\}$, where $r > 0$, but not at α itself.

2. Let f be a function with domain A, and suppose that α is a limit point of A. The function f **tends to infinity as z tends to α** if,

for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$f(z_n) \to \infty$

(or, equivalently, if,

for each positive number M, there is a positive number δ such that

$|f(z)| \geq M$, for $z \in A$ and $0 < |z - \alpha| < \delta$).

We write $f(z) \to \infty$ as $z \to \alpha$.

3. Suppose that a function f has a singularity at α. Then

(a) f has a **removable singularity** at α if there is a function g, analytic at α, and a positive number r, such that

$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r;$$

(b) f has a **pole of order k** at α if there is a function g, analytic at α with $g(\alpha) \neq 0$, and a positive number r, such that

$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r;$$

(c) f has an **essential singularity** at α if the singularity at α is neither removable nor a pole.

In case (a), g is an **analytic extension** of f to $\{z : |z - \alpha| < r\}$.

4. If a function f has a singularity at α and
1. $\lim_{z \to \alpha} f(z)$ does not exist,
2. $f(z)$ does not tend to infinity as $z \to \alpha$,

then α is an essential singularity of f.

Section 2: Laurent's Theorem

1. Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n = \cdots + \frac{a_{-2}}{(z - \alpha)^2} + \frac{a_{-1}}{(z - \alpha)} + a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots, \qquad (*)$$

where $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, is called an **extended power series about α**.

2. For a given z, the extended power series $(*)$ is convergent if $\sum_{n=0}^{\infty} a_n(z - \alpha)^n$, the **analytic part**, and $\sum_{n=1}^{\infty} a_{-n}(z - \alpha)^{-n}$, the **singular part**, are both convergent; its **sum** is found by adding the sums of these two series.

3. Let $A = \{z : \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \text{ converges}\}$. The function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \quad (z \in A)$$

is called the **sum function** of the extended power series.

4. If the analytic part of an extended power series has disc of convergence $\{z : |z - \alpha| < r\}$ and the singular part converges on $\{z : |z - \alpha| > 1/r'\}$, then the extended power series has **annulus of convergence**

$$A = \{z : |z - \alpha| < r\} \cap \{z : |z - \alpha| > 1/r'\}.$$

Depending on the values of r and r', A may be an open annulus, a punctured open disc, a punctured plane, the 'outside' of a closed disc, or the empty set.

5. Laurent's Theorem
If f is a function which is analytic on the open annulus

$$A = \{z : r_1 < |z - \alpha| < r_2\},$$

where $0 \leq r_1 < r_2 \leq \infty$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A, \qquad (*)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}}\, dz, \quad \text{for } n \in \mathbb{Z},$$

C being any circle with centre α lying in A.
Moreover, the representation $(*)$ of f is unique on A.

6. The representation $(*)$ is the **Laurent series (about α) for the function f on A.**
If A is a punctured open disc, then the representation $(*)$ is called the **Laurent series about α for f.**

7. A function f may have different Laurent series about α on different annuli.

8. If a function f has a singularity at α and if its Laurent series about α is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n,$$

then

(a) f has a removable singularity at α if and only if $a_n = 0$, for all $n < 0$;

(b) f has a pole of order k at α if and only if $a_n = 0$, for all $n < -k$, and $a_{-k} \neq 0$;

(c) f has an essential singularity at α if and only if $a_n \neq 0$, for infinitely many $n < 0$.

9. A Laurent series may be obtained from known Taylor series (for example, by using substitution) and by using partial fractions.

Section 3: Behaviour near a Singularity

1. Let a function f have a singularity at α. Then the following statements are equivalent:

   (A) f has a removable singularity at α;

   (B) $\lim_{z \to \alpha} f(z)$ exists;

   (C) f is bounded on $\{z : 0 < |z - \alpha| < r\}$, for some $r > 0$;

   (D) $\lim_{z \to \alpha} (z - \alpha) f(z) = 0$.

2. Let a function f have a singularity at α. Then the following statements are equivalent:

   (A) f has a pole of order k at α;

   (B) $\lim_{z \to \alpha} (z - \alpha)^k f(z)$ exists, and is non-zero;

   (C) $1/f$ has a removable singularity at α which, when removed, gives rise to a zero of order k at α.

3. Let a function f have a singularity at α. Then f has a pole at α if and only if

   $f(z) \to \infty$ as $z \to \alpha$.

4. **Casorati–Weierstrass Theorem**

   Suppose that a function f has an essential singularity at α. Let w be any complex number, and let ε and δ be positive real numbers. Then there exists $z \in \mathbb{C}$ such that

   $$0 < |z - \alpha| < \delta \quad \text{and} \quad |f(z) - w| < \varepsilon.$$

Section 4: Evaluating Residues using Laurent Series

1. If a function f is analytic on $D = \{z : 0 < |z - \alpha| < r\}$, then, by Laurent's Theorem,

   $$\int_C \frac{f(w)}{(w - \alpha)^{n+1}} \, dw = 2\pi i a_n,$$

   where $\sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n$ is the Laurent series about α for f, and C is any circle with centre α lying in D.

2. If a function f is analytic on a punctured disc D with centre α, then the coefficient a_{-1} in the Laurent series about α for f is called the **residue of f at α**, and is denoted by $\operatorname{Res}(f, \alpha)$. Hence

   $$\int_C f(z) \, dz = 2\pi i \operatorname{Res}(f, \alpha),$$

   where C is any circle with centre α lying in D.

PART II: INDEX

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Mathematics: A Third Level Course

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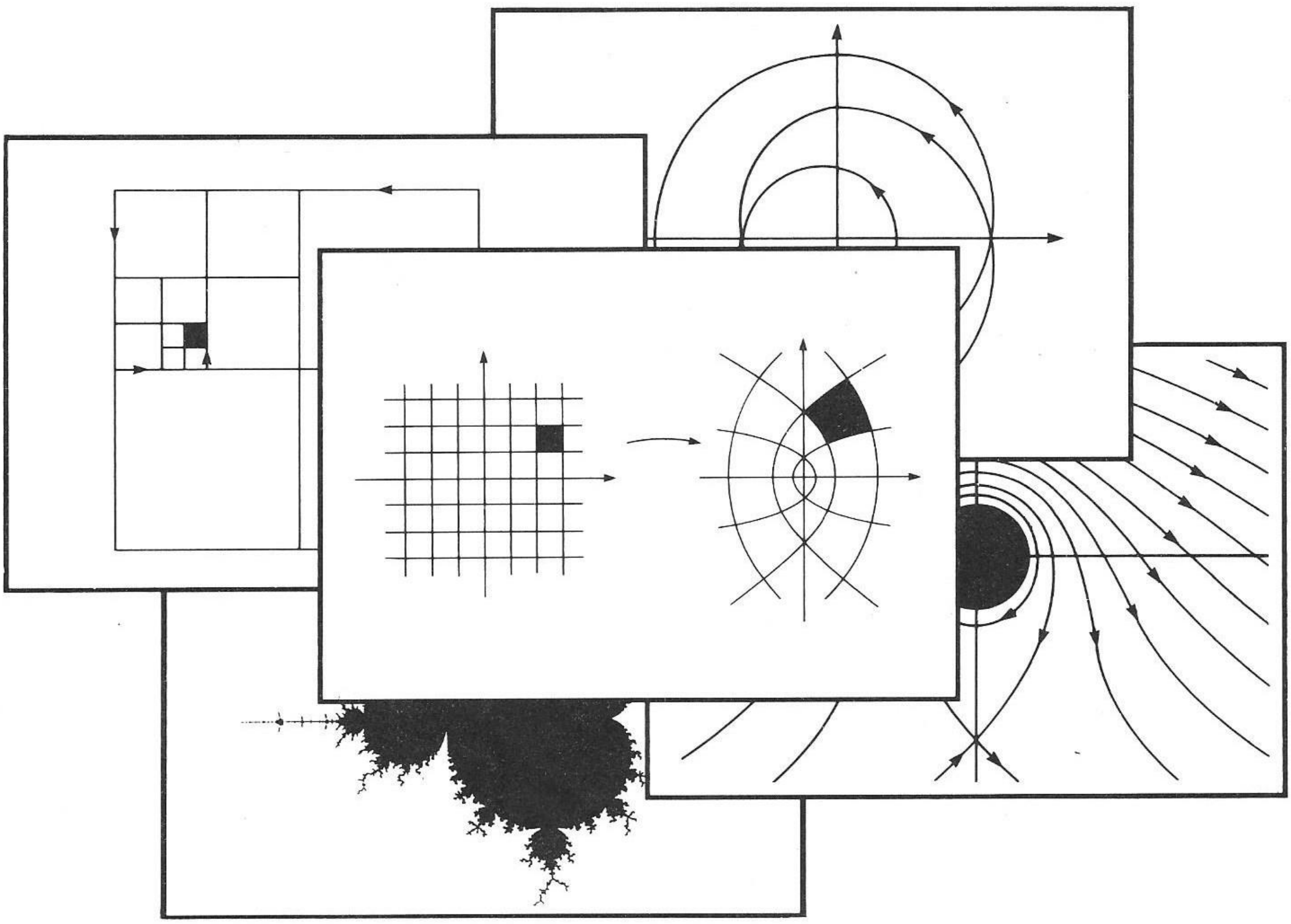
COMPLEX ANALYSIS

HANDBOOK THREE



COMPLEX ANALYSIS COMPLEX ANALYSIS COMPL

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CONTENTS

PART I: UNIT SUMMARIES

Unit C1 Residues

Section 1: Calculating Residues

1. Suppose that an analytic function f has a singularity at the point α. Then

$$\operatorname{Res}(f,\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z),$$

provided that this limit exists.

2. **g/h Rule**

Let $f(z) = g(z)/h(z)$, where the functions g and h are analytic at the point α, $h(\alpha) = 0$, and $h'(\alpha) \neq 0$. Then

$$\operatorname{Res}(f,\alpha) = g(\alpha)/h'(\alpha).$$

3. **Cover-up Rule**

Let $f(z) = \dfrac{p(z)}{(z-\alpha)q(z)}$, where the functions p and q are analytic at the point α, and $q(\alpha) \neq 0$. Then

$$\operatorname{Res}(f,\alpha) = p(\alpha)/q(\alpha).$$

4. If a function f has a pole of order k at the point α, then the residue of f at α is given by

$$\operatorname{Res}(f,\alpha) = \frac{1}{(k-1)!}\lim_{z\to\alpha}\left(\frac{d^{k-1}}{dz^{k-1}}((z-\alpha)^k f(z))\right).$$

Section 2: The Residue Theorem

1. **Cauchy's Residue Theorem**

Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$ except for a finite number of singularities. Let Γ be any simple-closed contour in $\mathcal{R}$, not passing through any of these singularities. Then

$$\int_\Gamma f(z)\,dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ.

2. **Strategy for evaluating $\displaystyle\int_0^{2\pi}\Phi(\cos t,\sin t)\,dt$**

(a) Replace

$$\cos t \text{ by } \frac{1}{2}\left(z+z^{-1}\right), \sin t \text{ by } \frac{1}{2i}\left(z-z^{-1}\right)$$

$$\text{and } dt \text{ by } \frac{1}{iz}dz,$$

to obtain a contour integral of the form $\int_C f(z)\,dz$ around the unit circle $C = \{z : |z| = 1\}$.

(b) Locate the singularities of the function f lying inside C, and calculate the residues of f at these points.

(c) Evaluate the given integral by calculating

$$2\pi i \times (\text{the sum of the residues found in step (b)}).$$

Section 3: Evaluating Improper Integrals

1. Let f be a function defined on an unbounded interval $]a,\infty[$. Then the function f has **limit α as r tends to ∞** if

for each sequence $\{r_n\}$ in $]a,\infty[$ such that $r_n \to \infty$, $f(r_n) \to \alpha$ as $n \to \infty$.

In this case we write

EITHER $\displaystyle\lim_{r\to\infty} f(r) = \alpha$,

OR $f(r) \to \alpha$ as $r \to \infty$.

2. **Combination Rules**

Let f and g be functions such that

$$\lim_{r\to\infty} f(r) = \alpha \quad\text{and}\quad \lim_{r\to\infty} g(r) = \beta.$$

Then

Sum Rule $\displaystyle\lim_{r\to\infty}(f(r)+g(r)) = \alpha+\beta$;

Multiple Rule $\displaystyle\lim_{r\to\infty}(\lambda f(r)) = \lambda\alpha$, for $\lambda \in \mathbb{C}$;

Product Rule $\displaystyle\lim_{r\to\infty}(f(r)g(r)) = \alpha\beta$;

Quotient Rule $\displaystyle\lim_{r\to\infty}(f(r)/g(r)) = \alpha/\beta$, provided that $\beta \neq 0$.

3. If p and q are polynomial functions such that the degree of q exceeds the degree of p, then

$$\lim_{r\to\infty}\frac{p(r)}{q(r)} = 0.$$

4. Let f be a continuous function with domain $\mathbb{R}$. Then the **improper integral** $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r\to\infty}\int_{-r}^{r} f(t)\,dt,$$

provided that this limit exists.

Let f be a function which is continuous on the interval $[a,\infty[$. Then the **improper integral** $\displaystyle\int_a^\infty f(t)\,dt$ is

$$\int_a^\infty f(t)\,dt = \lim_{r\to\infty}\int_a^r f(t)\,dt,$$

provided that this limit exists.

5. Let f be a continuous function with domain $\mathbb{R}$. Then:

(a) if f is an odd function, $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 0$;

(b) if f is an even function,

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\int_0^\infty f(t)\,dt,$$

provided that these integrals exist.

6. Let a function f be defined and continuous at all points of an interval $[a, b]$ except at the point $c \in]a, b[$. Then the **improper integral** $\int_a^b f(t)\,dt$ is

$$\int_a^b f(t)\,dt = \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^b f(t)\,dt \right),$$

where the limit is taken as ε tends to 0 through *positive* values, provided that this limit exists.

Let a function f be continuous at all points of $\mathbb{R}$, except at the point c. Then the **improper integral** $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r \to \infty} \int_{-r}^{r} f(t)\,dt$$

$$= \lim_{r \to \infty} \left(\lim_{\varepsilon \to 0} \left(\int_{-r}^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^{r} f(t)\,dt \right) \right),$$

provided that these limits exist.

7. Round-the-Pole Lemma

If a function f has a simple pole at α and Γ is the semicircular contour (traversed anticlockwise) from $\alpha + \varepsilon$ to $\alpha - \varepsilon$, then

$$\lim_{\varepsilon \to 0} \int_\Gamma f(z)\,dz = \pi i \operatorname{Res}(f, \alpha).$$

8. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least two;
2. any poles of p/q on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function p/q at those poles in the upper half-plane, and T is the sum of the residues of the function p/q at those poles on the real axis.

9. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least one;
2. any poles of p/q on the real axis are simple.

Then, if $k > 0$,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\,dt = 2\pi i S + \pi i T, \qquad (*)$$

where S is the sum of the residues of the function $z \longmapsto (p(z)/q(z))e^{ikz}$ at those poles in the upper half-plane, and T is the sum of the residues of the function $z \longmapsto (p(z)/q(z))e^{ikz}$ at those poles on the real axis.

By equating the real parts and imaginary parts of Equation $(*)$, we obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\,dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\,dt.$$

10. Jordan's Lemma

Let Γ be the semicircular contour (traversed anticlockwise) from r to $-r$ and suppose that a function f is continuous on Γ and satisfies

$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$

Then, for $k > 0$, we have

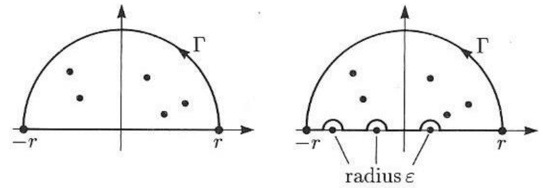
$$\left| \int_\Gamma f(z) e^{ikz}\,dz \right| \le \frac{M\pi}{k}.$$

11. Improper integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt, \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\,dt \text{ and } \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\,dt,$$

which are the subject of the results in items 8 and 9, may also be evaluated by means of a five-step method:

(a) Consider an appropriate contour integral I, where the contour Γ is of one of the following forms.



(b) Evaluate I by using the Residue Theorem.

(c) Split up I into line segments and semicircles.

(d) Estimate the integral along the semicircle of radius r.

(e) Let $r \to \infty$, $\varepsilon \to 0$ (if appropriate) in the equation resulting from step (c), and then deduce the value of the improper integral.

Section 4: Summing Series

1. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = \pi \cot \pi z \cdot \phi(z)$ is such that

$$\lim_{N \to \infty} \int_{S_N} f(z)\,dz = 0.$$

Then

$$\sum_{n=1}^{\infty} \phi(n) = -\frac{1}{2} \left(\operatorname{Res}(f, 0) + \sum_{j=1}^{k} \operatorname{Res}(f, \alpha_j) \right).$$

Further, if ϕ is analytic at 0, then

$$\operatorname{Res}(f, 0) = \phi(0).$$

2. For each $N = 1, 2, \ldots,$

$$|\cot \pi z| \le 2, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$.

3. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z)$ is such that

$$\lim_{N\to\infty} \int_{S_N} f(z)\, dz = 0.$$

Then

$$\sum_{n=1}^{\infty} (-1)^n \phi(n) = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right).$$

Further, if ϕ is analytic at 0, then

$$\operatorname{Res}(f,0) = \phi(0).$$

4. For each $N = 1, 2, \ldots,$

$$|\operatorname{cosec} \pi z| \le 1, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm 1 \pm i)$.

5. The Laurent series about 0 for cot and cosec are:

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \cdots;$$

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots.$$

Unit C2 Zeros and Extrema

Section 1: The Winding Number

1. A **continuous argument function** for a path $\Gamma : \gamma(t)$ $(t \in [a,b])$ lying in $\mathbb{C} - \{0\}$ is a continuous function

$$\theta : [a,b] \longmapsto \mathbb{R}$$

such that, for each $t \in [a,b]$, $\theta(t)$ is an argument of $\gamma(t)$.

Such a function θ satisfies

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \quad \text{for } t \in [a,b].$$

2. Any path $\Gamma : \gamma(t)$ $(t \in [a,b])$ lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

3. The **winding number** of a path $\Gamma : \gamma(t)$ $(t \in [a,b])$ lying in $\mathbb{C} - \{0\}$ **round 0** is

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any continuous argument function for Γ.

$\operatorname{Wnd}(\Gamma, 0)$ may be calculated by inspecting a sketch of Γ, if one is available.

4. For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by

$$\operatorname{Arg}_\phi(z) = \theta \quad (z \in \mathbb{C} - \{0\}),$$

where θ is the argument of z lying in the interval $]\phi - 2\pi, \phi]$.

5. For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on the cut plane

$$\mathbb{C}_\phi = \left\{re^{i\theta} : r > 0, \phi - 2\pi < \theta < \phi\right\}.$$

6. For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by

$$\operatorname{Log}_\phi(z) = \log_e |z| + i \operatorname{Arg}_\phi(z) \quad (z \in \mathbb{C} - \{0\}).$$

7. For all $\phi \in \mathbb{R}$, Log_ϕ is analytic on $\mathbb{C}_\phi$ with derivative Log'_ϕ given by

$$\operatorname{Log}'_\phi(z) = 1/z, \quad \text{for } z \in \mathbb{C}_\phi.$$

8. Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a closed contour lying in $\mathbb{C} - \{0\}$. Then

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\, dz.$$

9. The **winding number** of the path $\Gamma : \gamma(t)$ $(t \in [a,b])$ lying in $\mathbb{C} - \{\alpha\}$ **round** α is

$$\operatorname{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi}(\theta_\alpha(b) - \theta_\alpha(a)),$$

where θ_α is a continuous argument function for Γ relative to α (that is, θ_α is continuous on $[a,b]$ and $\theta_\alpha(t)$ is an argument of $\gamma(t) - \alpha$).

10. $\mathrm{Wnd}(\Gamma, \alpha) = \mathrm{Wnd}(\Gamma - \alpha, 0)$,

where

$\Gamma - \alpha : \gamma(t) - \alpha \quad (t \in [a, b])$

is the path Γ translated by $-\alpha$.

11. Let Γ be a closed path and let D be an open disc lying in the complement of Γ. Then the function $\alpha \longmapsto \mathrm{Wnd}(\Gamma, \alpha)$ is constant on D.

Section 2: Locating Zeros of Analytic Functions

1. Let an analytic function f have a zero of order n at α. Then the function f'/f has a simple pole at α with

$\mathrm{Res}(f'/f, \alpha) = n.$

2. **Argument Principle**

Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$\mathrm{Wnd}(f(\Gamma), 0) = N,$

where N is the number of zeros of f inside Γ, counted according to their orders.

3. Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z) \neq \beta$, for $z \in \Gamma$. Then $\mathrm{Wnd}(\Gamma, \beta)$ is the number of zeros of the function $f - \beta$ inside Γ, counted according to their orders.

Unless otherwise stated, zeros are always counted according to their orders.

4. **Rouché's Theorem**

Suppose that

1. the function f is analytic on a simply-connected region $\mathcal{R}$;
2. Γ is a simple-closed contour in $\mathcal{R}$ and

   $|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$

Then f has the same number of zeros as g inside Γ. (The function g is known as a 'dominant term' in f on Γ.)

5. **Fundamental Theorem of Algebra**

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and $a_n \neq 0$. Then p has exactly n zeros, all lying in the open disc $\{z : |z| < R\}$, where

$R = 1 + \max\{|a_0|/|a_n|, \ldots, |a_{n-1}|/|a_n|\}.$

Section 3: Local Behaviour of Analytic Functions

1. **Open Mapping Theorem**

Let the function f be analytic and non-constant on a region $\mathcal{R}$ and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

2. If a function f is analytic and non-constant on a region $\mathcal{R}$, then $f(\mathcal{R})$ is a region.

3. Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then f is **n-one near α** if there is a region $\mathcal{S}$ in $\mathcal{R}$, with $\alpha \in \mathcal{S}$, and a function ϕ which is analytic and one-one on its domain $\mathcal{S}$, such that

$f(z) = f(\alpha) + (\phi(z))^n, \quad \text{for } z \in \mathcal{S}.$

4. **Local Mapping Theorem**

Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots, \quad (*)$$

where $n \geq 1$ and $a_n \neq 0$, if and only if f is n-one near α.

5. The Taylor series for f about α takes the form $(*)$ if and only if

$0 = f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha),$

but $f^{(n)}(\alpha) \neq 0$.

6. **Inverse Function Rule**

Let f be a one-one analytic function whose domain is a region $\mathcal{R}$. Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$\left(f^{-1}\right)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \quad \text{for } \beta \in f(\mathcal{R}).$$

7. The inverse functions $\tan^{-1}$ and $\sin^{-1}$ are analytic.

8. **Strategy for inverting a Taylor series**

Given the Taylor series about α for f:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1}:

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w-\beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z - \alpha)$ in the identity

$$\begin{aligned} z - \alpha &= b_1(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots) \\ &\quad + b_2(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots)^2 + \cdots, \end{aligned}$$

to obtain equations for $b_1, b_2, \ldots$, in terms of $a_1, a_2, \ldots$.

Section 4: Extreme Values of Analytic Functions

1. Let a function f be defined on a region $\mathcal{R}$. Then the function $|f|$ has a **local maximum** at the point $\alpha \in \mathcal{R}$ if there is some $r > 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z - \alpha| < r.$$

2. Local Maximum Principle

Let a function f be analytic on a region $\mathcal{R}$. If f is non-constant on $\mathcal{R}$, then the function $|f|$ has no local maxima on $\mathcal{R}$.

3. The **closure** $\overline{A}$ of a set A in $\mathbb{C}$ is

$$\overline{A} = \operatorname{int} A \cup \partial A.$$

4. Maximum Principle

Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

5. Minimum Principle

Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \geq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

6. Boundary Uniqueness Theorem

Let functions f and g be analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial\mathcal{R}$, then $f = g$ on $\mathcal{R}$.

7. Schwarz's Lemma

Let a function f be analytic on $\{z : |z| < R\}$ with $f(0) = 0$, and suppose that

$$|f(z)| \leq M, \quad \text{for } |z| < R.$$

Then

$$|f(z)| \leq (M/R)|z|, \quad \text{for } |z| < R.$$

Unit C3 Analytic Continuation

Section 1: What is Analytic Continuation?

1. Let f and g be analytic functions whose domains are the regions $\mathcal{R}$ and $\mathcal{S}$, respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T}.$$

We also say that g is a **direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$**, and vice versa.

2. Let a function f be continuous on the interval $]0, \infty[$. Then

$$\int_0^\infty f(t)\,dt = \lim_{\varepsilon \to 0} \int_\varepsilon^1 f(t)\,dt + \lim_{r \to \infty} \int_1^r f(t)\,dt,$$

provided that both these limits exist.

3. Improper integrals of the forms

$$\int_0^\infty \frac{p(t)}{q(t)} \log_e t\,dt, \quad \int_0^\infty \frac{p(t)}{q(t)} t^a\,dt, \quad \int_0^\infty \frac{p(t)}{q(t)}\,dt,$$

where $0 < a < 1$ and p, q are polynomial functions such that p/q is even, the degree of q exceeds that of p by at least two and any poles of p/q on the non-negative real axis are simple, may be evaluated by a five-step method like that given in item 11 of *Unit C1*, Section 3 above.

4. Let p and q be polynomial functions such that

1. the degree of q exceeds the degree of p by at least 2;
2. any poles of p/q on the non-negative real axis are simple.

Then, for $0 < a < 1$,

$$\int_0^\infty \frac{p(t)}{q(t)} t^a\,dt = -\left(\pi e^{-\pi a i} \operatorname{cosec} \pi a\right) S - (\pi \cot \pi a) T,$$

where S is the sum of the residues of the function

$$f_1(z) = \frac{p(z)}{q(z)} \exp\left(a \operatorname{Log}_{2\pi}(z)\right)$$

in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

$$f_2(z) = \frac{p(z)}{q(z)} \exp\left(a \operatorname{Log} z\right)$$

on the positive real axis.

Section 2: Indirect Analytic Continuation

1. Let an analytic function f have as domain a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. If the disc of convergence D of the Taylor series about α for f contains points which are not in $\mathcal{R}$, then the function g, with domain D, defined by the Taylor series is a **direct analytic continuation of f by Taylor series.**

2. For convenience an analytic function f whose domain is a region $\mathcal{R}$ is sometimes written $(f, \mathcal{R})$.

3. The finite sequence of functions

$$(f_1, \mathcal{R}_1), (f_2, \mathcal{R}_2), \ldots, (f_n, \mathcal{R}_n)$$

forms a **chain** if, for $k = 1, 2, \ldots, n-1$,

$(f_{k+1}, \mathcal{R}_{k+1})$ is a direct analytic continuation of $(f_k, \mathcal{R}_k)$.

Any two functions of the chain are called **analytic continuations of each other**, and the chain is said to **join** $(f_1, \mathcal{R}_1)$ to $(f_n, \mathcal{R}_n)$. If $\mathcal{R}_1 = \mathcal{R}_n$, then the chain is said to be **closed**.

Two functions of a chain which are not direct analytic continuations of each other are called **indirect analytic continuations.**

Section 3: Uniform Convergence

1. A sequence of functions $\{f_n\}$ **converges pointwise** (to a **limit function f**) **on a set E** if, for each $z \in E$,

$$\lim_{n \to \infty} f_n(z) = f(z).$$

2. A sequence of functions $\{f_n\}$ **converges uniformly** (to a **limit function f**) **on a set E** if

for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \text{ for all } n > N, \text{ and all } z \in E.$$

We also say that $\{f_n\}$ is **uniformly convergent on E, with limit function f.**

3. **Strategy for proving uniform convergence**

To prove that a sequence of functions $\{f_n\}$ converges uniformly on a set E:

(a) determine the limit function f by evaluating

$$f(z) = \lim_{n \to \infty} f_n(z), \quad \text{for } z \in E;$$

(b) find a null sequence $\{a_n\}$ of positive terms such that

$$|f_n(z) - f(z)| \le a_n, \quad \text{for } n = 1, 2, \ldots, \text{ and all } z \in E.$$

4. If $\{\phi_n\}$ is a sequence of functions, then the series of functions

$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \cdots$$

converges pointwise/uniformly on a set E if the sequence of **partial sum functions**

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z)$$

converges pointwise/uniformly on E, respectively. The limit function f of the sequence $\{f_n\}$ is called the **sum function** of $\sum_{n=1}^{\infty} \phi_n$ **on E**, written

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E).$$

5. **Weierstrass' M-test**

Let $\{\phi_n\}$ be a sequence of functions defined on a set E and suppose that there is a sequence of positive terms $\{M_n\}$, such that

1. $|\phi_n(z)| \le M_n$, for $n = 1, 2, \ldots$, and all $z \in E$;
2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E.

6. **Weierstrass' Theorem**

Let $\{f_n\}$ be a sequence of functions which are analytic on a region $\mathcal{R}$ and which converge uniformly to a function f on each closed disc in $\mathcal{R}$. Then

f is analytic on $\mathcal{R}$

and

the sequence $\{f_n'\}$ converges uniformly to f' on each closed disc in $\mathcal{R}$.

7. When Weierstrass' Theorem is used to prove that a function f defined by a series is analytic, the derivative f' may be obtained by term by term differentiation of the series.

8. The **zeta function** ζ, defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (\operatorname{Re} z > 1),$$

is analytic, and can be continued analytically to $\mathbb{C} - \{1\}$.

Section 4: The Gamma Function

1. Let $\mathcal{R}$ be a region and let K be a complex-valued function of the two variables $z \in \mathcal{R}$ and $t \in [a, b]$, such that

1. K is analytic on $\mathcal{R}$ as a function of z, for each $t \in [a, b]$;
2. K and $\partial K/\partial z$ are continuous on $[a, b]$ as functions of t, for each $z \in \mathcal{R}$;
3. for some $M > 0$,
$$|K(z,t)| \leq M, \quad \text{for } z \in \mathcal{R}, t \in [a, b].$$

Then the function
$$f(z) = \int_a^b K(z,t)\,dt \quad (z \in \mathcal{R})$$
is analytic on $\mathcal{R}$ and
$$f'(z) = \int_a^b \frac{\partial K}{\partial z}(z,t)\,dt, \quad \text{for } z \in \mathcal{R}.$$

2. The gamma function Γ, defined by
$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt,$$
is such that

(a) Γ is analytic on $H = \{z : \operatorname{Re} z > 1\}$,

(b) $\Gamma(n) = (n-1)!$, for $n > 1$,

(c) $\Gamma(z+1) = z\Gamma(z)$, for $z \in H$.

The identity in part (c) is called the **functional equation of the gamma function.**

3. The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0, -1, -2, \ldots\}$ with simple poles at $0, -1, -2, \ldots,$ such that
$$\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$
The functional equation of the gamma function holds on $\mathbb{C} - \{0, -1, -2, \ldots\}$.

4. The value of the gamma function at $\frac{3}{2}$ is
$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

5. An alternative definition of the gamma function is
$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)},$$
for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$.

Section 5: Riemann's Legacy

This a reading-only section.

PART II: INDEX

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Mathematics: A Third Level Course

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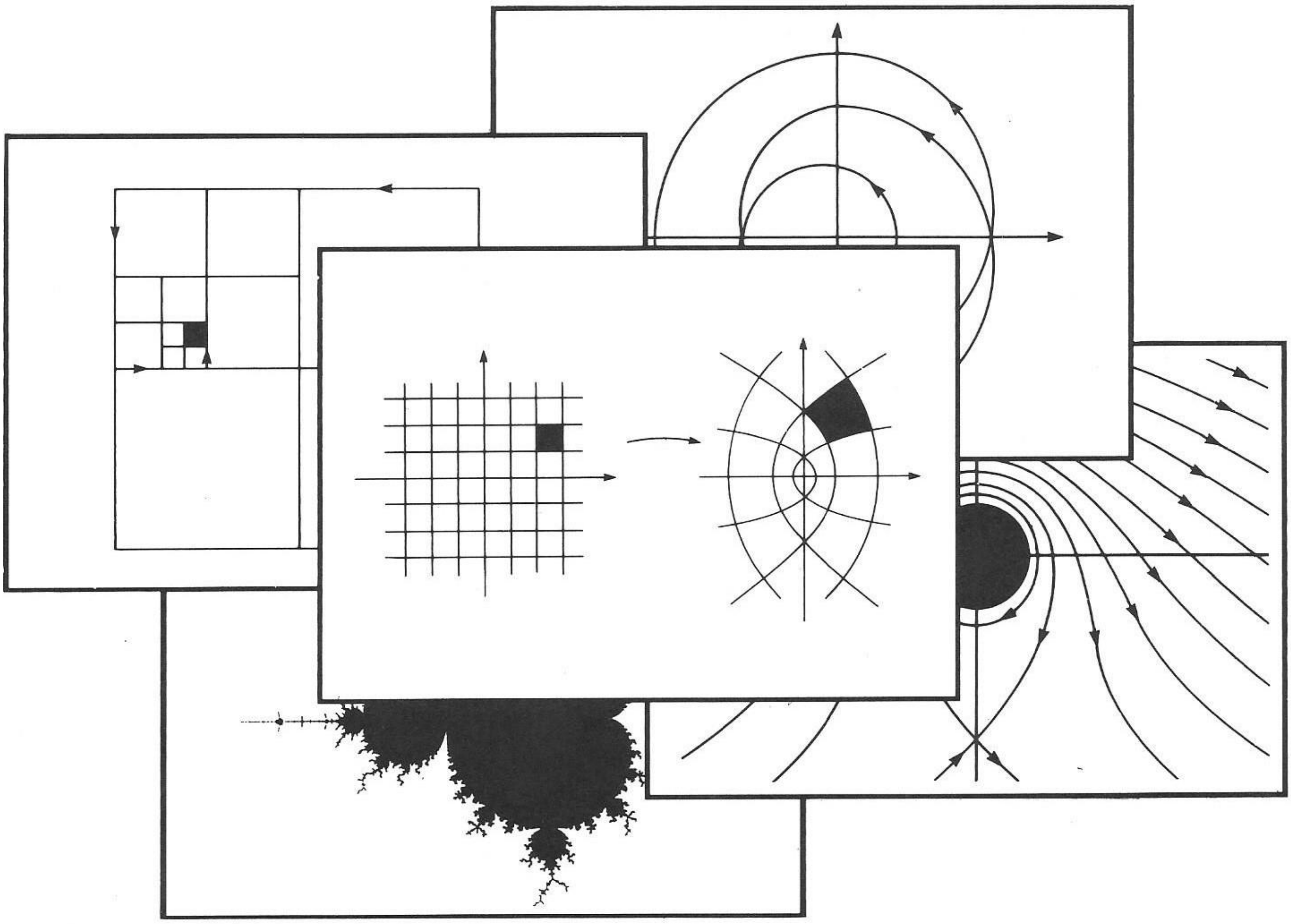
COMPLEX ANALYSIS

HANDBOOK FOUR



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CONTENTS

PART I: UNIT SUMMARIES

Unit D1 Conformal Mappings

Section 1: Linear and Reciprocal Functions

1. A function of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called a **linear function.**

2. Linear functions map circles onto circles and straight lines onto straight lines. Furthermore,

(a) given any two circles C_1 and C_2, there is a linear function that maps C_1 onto C_2;

(b) given any two lines L_1 and L_2, there is a linear function that maps L_1 onto L_2.

3. The function $f(z) = 1/z$ is called the **reciprocal function.**

4. **Strategy for finding the equation of the image of a path under $f(z) = 1/z$**

To find the equation of the image $f(\Gamma)$ of a path Γ under $f(z) = 1/z$:

(a) write down an equation that relates the x- and y-coordinates of all points $x + iy$ on Γ;

(b) replace x by $\dfrac{u}{u^2+v^2}$ and y by $\dfrac{-v}{u^2+v^2}$;

(c) simplify the resulting equation to obtain an equation that relates the u- and v-coordinates of all points $u + iv$ on the image $f(\Gamma)$.

5. Every line or circle has an equation of the form

$$a\left(x^2+y^2\right) + bx + cy + d = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2 + c^2 > 4ad$.
Conversely, any such equation represents a line or a circle.

Also

(a) the equation represents a line if and only if $a = 0$;

(b) the line or circle passes through the origin if and only if $d = 0$.

6. The **extended complex plane** $\widehat{\mathbb{C}}$ is the union of the ordinary complex plane $\mathbb{C}$ and one extra point, the **point at infinity**, which is denoted by ∞; thus $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

7. A function f has a **removable singularity**, **a pole of order k**, or an **essential singularity at ∞**, if the function $g(w) = f(1/w)$ has the corresponding type of singularity at 0.

8. Let f be a rational function. The **extended function** $\widehat{f}$ associated with f is the extension of f to $\widehat{\mathbb{C}}$ obtained by defining, for each singularity α of f in $\widehat{\mathbb{C}}$:

$$\widehat{f}(\alpha) = \beta,$$

where β in $\widehat{\mathbb{C}}$ satisfies

$$f(z) \to \beta \text{ as } z \to \alpha.$$

9. If L is a line in $\mathbb{C}$, then $L \cup \{\infty\}$ is an **extended line** in $\widehat{\mathbb{C}}$.

10. A **generalized circle** is either a circle or an extended line. It is completely determined by three distinct points on it.

11. Extended linear functions and the extended reciprocal function map

(a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

(b) generalized circles onto generalized circles.

12. The sphere $\mathbb{S}$ of unit radius and centre the origin is called the **Riemann sphere.**

13. The function $\pi : \mathbb{S} \longrightarrow \widehat{\mathbb{C}}$, defined by

$$\pi(u, v, s) = \begin{cases} \left(\dfrac{u}{1-s}\right) + \left(\dfrac{v}{1-s}\right)i, & (u, v, s) \in \mathbb{S} - \{(0,0,1)\}, \\ \infty, & (u, v, s) = (0,0,1), \end{cases}$$

is called **stereographic projection.**

14. (a) Under stereographic projection, all ordinary circles on the Riemann sphere $\mathbb{S}$ map to generalized circles in $\widehat{\mathbb{C}}$.

(b) Stereographic projection preserves angles.

Section 2: Möbius Transformations

1. A function of the form

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is called a **Möbius transformation.**

2. Every Möbius transformation is analytic and conformal.

3. The extended function associated with a Möbius transformation is called the **extended Möbius transformation.**

4. Every Möbius transformation is either a linear function, or a composition of linear functions and the reciprocal function.

5. Extended Möbius transformations map

(a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

(b) generalized circles onto generalized circles.

6. **Inverse function of a Möbius transformation**

The Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is a one-one function from $\mathbb{C} - \{-d/c\}$ onto $\mathbb{C} - \{a/c\}$, with inverse function

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Furthermore, $\widehat{f}^{-1} = \widehat{f^{-1}}$.

7. The set of extended Möbius transformations has the following group properties.

Closure If $\widehat{f}$ and $\widehat{g}$ are extended Möbius transformations, then so is $\widehat{f} \circ \widehat{g}$.

Identity The identity function on $\widehat{\mathbb{C}}$ is an extended Möbius transformation.

Inverses Every extended Möbius transformation $\widehat{f}$ has an inverse function $\widehat{f}^{-1}$.

Associativity If $\widehat{f}$, $\widehat{g}$ and $\widehat{h}$ are extended Möbius transformations, then

$$\widehat{f} \circ (\widehat{g} \circ \widehat{h}) = (\widehat{f} \circ \widehat{g}) \circ \widehat{h}.$$

8. A point α in the extended complex plane $\widehat{\mathbb{C}}$ is a **fixed point** of an extended Möbius transformation $\widehat{f}$ if $\widehat{f}(\alpha) = \alpha$.

9. An extended Möbius transformation, other than the identity function, has either one or two fixed points in $\widehat{\mathbb{C}}$.

10. If $\widehat{f}$ and $\widehat{g}$ are two extended Möbius transformations that agree on a set consisting of three or more points in $\widehat{\mathbb{C}}$, then $\widehat{f} = \widehat{g}$.

11. Given three distinct points α, β, γ in $\widehat{\mathbb{C}}$, and any other three distinct points α', β', γ' in $\widehat{\mathbb{C}}$, there is a unique extended Möbius transformation $\widehat{f}$ that maps

$$\alpha \text{ to } \alpha', \quad \beta \text{ to } \beta' \quad \text{and} \quad \gamma \text{ to } \gamma'.$$

In particular, if α', β', γ' is the **standard triple of points** 0, 1, ∞, respectively, then $\widehat{f}$ corresponds to

$$f(z) = \frac{(z-\alpha)}{(z-\gamma)} \frac{(\beta-\gamma)}{(\beta-\alpha)}.$$

In general, f is given by the **Implicit Formula**

$$\frac{(z-\alpha)}{(z-\gamma)} \frac{(\beta-\gamma)}{(\beta-\alpha)} = \frac{(w-\alpha')}{(w-\gamma')} \frac{(\beta'-\gamma')}{(\beta'-\alpha')}.$$

12. If C_1 and C_2 are generalized circles, then there is an extended Möbius transformation that maps C_1 onto C_2.

Section 3: Images of Generalized Circles

1. Although the images under an extended Möbius transformation of three distinct points on a generalized circle C do determine the image of C, it may be difficult to identify that image.

2. The **Apollonian form** of the equation of a generalized circle is

$$|z-\alpha| = k|z-\beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0.$$

If $k = 1$, the generalized circle is an extended line.

3. Let C be a generalized circle. Then α and β are **inverse points with respect to** C if

EITHER α and β are equal and lie on C;

OR there exists an extended Möbius transformation $\widehat{f}$ that maps α to 0, β to ∞, and C to the unit circle.

4. The points α and β in $\widehat{\mathbb{C}}$ are distinct inverse points with respect to a generalized circle C if and only if

EITHER both α and $\beta \in \mathbb{C}$ and C has the equation

$$|z-\alpha| = k|z-\beta|, \quad \text{for some } k > 0;$$

OR one of the points (β say) is ∞ and C has the equation

$$|z-\alpha| = r, \quad \text{for some } r > 0.$$

5. If α is the centre of a circle C, then α and ∞ are inverse points with respect to C.

6. Let $\widehat{f}$ be an extended Möbius transformation. If α and β are inverse points with respect to the generalized circle C, then $\widehat{f}(\alpha)$ and $\widehat{f}(\beta)$ are inverse points with respect to $\widehat{f}(C)$.

7. Let C be the generalized circle with equation

$$|z-\alpha| = k|z-\beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0.$$

(a) If $k \neq 1$, then C is a circle with centre λ and radius r, where

$$\lambda = \frac{\alpha - k^2\beta}{1-k^2} \quad \text{and} \quad r = \frac{k|\alpha-\beta|}{|1-k^2|}.$$

Also, λ lies on the line through α and β, and

$$(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2. \qquad (*)$$

(b) If $k = 1$, then C is the line through $\frac{1}{2}(\alpha + \beta)$ perpendicular to $\beta - \alpha$.

8. The process of applying the function $f(z) = 1/\overline{z}$ is known as **inversion**.

9. Let C be a generalized circle and let β be an arbitrary point of $\widehat{\mathbb{C}}$. Then β has a unique inverse point α with respect to C, which can be found by applying Equation $(*)$ if C is a circle or by reflection if C is an extended line.

Section 4: Transforming Regions

1. An **open disc centred at ∞** is a set of the form
$$\{z : |z| > M\} \cup \{\infty\}.$$

2. Let A be a subset of $\widehat{\mathbb{C}}$ and let $\alpha \in \widehat{\mathbb{C}}$. Then α is a **boundary point in $\widehat{\mathbb{C}}$** of A if each open disc centred at α contains at least one point of A and at least one point of $\widehat{\mathbb{C}} - A$.

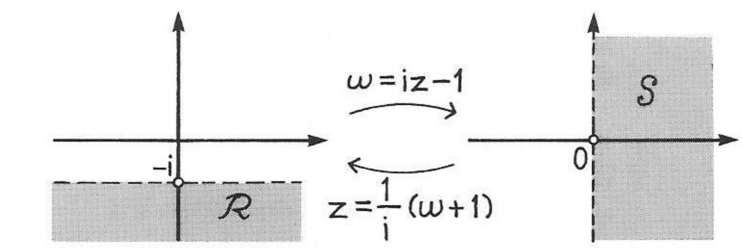
   The set of boundary points in $\widehat{\mathbb{C}}$ of A forms the **boundary in $\widehat{\mathbb{C}}$** of A.

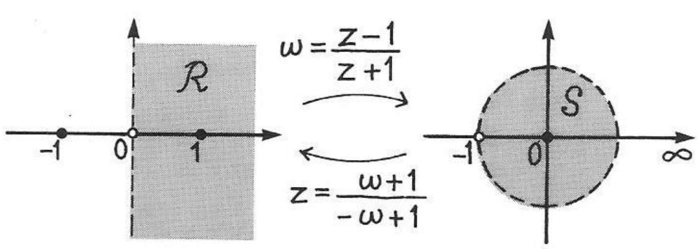
3. Let f be a Möbius transformation and let $\mathcal{R}$ be a region in the domain of f. Then $f(\mathcal{R})$ is a region and the extended Möbius transformation $\hat{f}$ maps the boundary in $\widehat{\mathbb{C}}$ of $\mathcal{R}$ onto the boundary in $\widehat{\mathbb{C}}$ of $f(\mathcal{R})$.

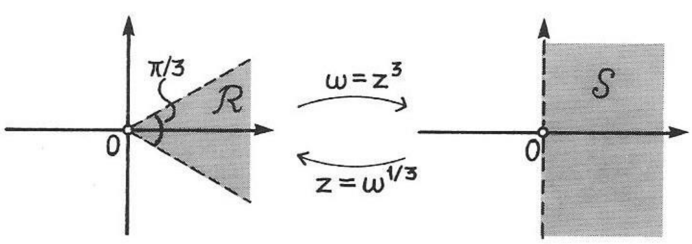
4. Any generalized circle divides $\widehat{\mathbb{C}}$ into two parts, each of which is called a **generalized open disc**.

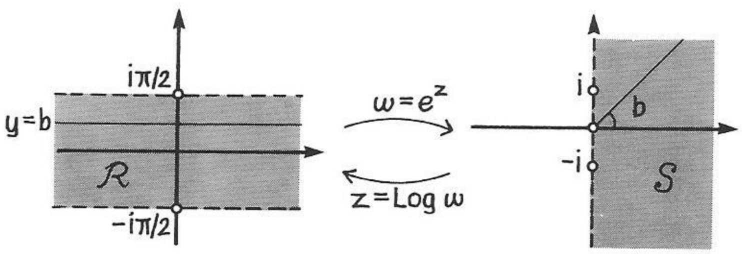
5. Conformal mappings between basic regions may be used to find, by composition, conformal mappings between other regions.

   Examples of one-one conformal mappings from $\mathcal{R}$ onto $\mathcal{S}$ and their inverse functions are given in the following figures.









6. The restriction of the function tan to
$$\mathcal{R} = \{z : -\pi/2 < \text{Re}\, z < \pi/2\}$$
has inverse function $\tan^{-1}$ defined by
$$\tan^{-1} w = \frac{1}{2i} \text{Log}\left(\frac{1+iw}{1-iw}\right) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{iv : |v| \geq 1\}$.

7. The restriction of the function sin to
$$\mathcal{R} = \{z : -\pi/2 < \text{Re}\, z < \pi/2\}$$
has inverse function $\sin^{-1}$ defined by
$$\sin^{-1} w = \frac{1}{i} \text{Log}(iw + \sqrt{1-w^2}) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{u \in \mathbb{R} : |u| \geq 1\}$.

8. The **Joukowski function** J is defined by
$$J(z) = z + \frac{1}{z} \quad (z \neq 0).$$
The restriction of J to $\{z : |z| > 1\}$ has inverse function J^{-1} defined by
$$J^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4/w^2}\right) \quad (w \in \mathbb{C} - [-2, 2]).$$

Section 5: The Riemann Mapping Theorem

This is a reading-only section.

Unit D2 Fluid Flows

Section 1: Setting up the Model

1. Basic Mathematical Model

We assume that

(i) the fluid forms a continuum, and any spatial variation of the flow velocity is continuous;

(ii) the flow is two-dimensional;

(iii) the flow is steady.

With these assumptions, we may represent the **flow velocity** at all times by a continuous complex function q, whose domain is the region occupied by the fluid.

2. If $q(z_0) = 0$, then z_0 is a **stagnation point** of the flow with velocity function q.

3. If q is a constant function, then the associated flow is a **uniform flow**.

4. A **streamline** through the point z_0, for a flow with velocity function q, is a smooth path $\Gamma : \gamma(t)\ (t \in I)$ such that

(a) $\gamma'(t) = q(\gamma(t))$, for $t \in I$;

(b) $z_0 = \gamma(t_0)$, for some $t_0 \in I$.

If $q(z_0) = 0$, then $\{z_0\}$ is a **degenerate streamline**, with constant parametrization

$$\gamma(t) = z_0 \quad (t \in I).$$

5. The component of $q(z)$ in the direction specified by $e^{i\theta}$ is

$$q_\theta(z) = \mathrm{Re}\left(\overline{q(z)}e^{i\theta}\right).$$

The component of $q(z)$ in the direction specified by $e^{i(\theta-\pi/2)}$ is

$$q_{(\theta-\pi/2)}(z) = \mathrm{Im}\left(\overline{q(z)}e^{i\theta}\right).$$

6. The conjugate velocity function $\overline{q}$ is given by

$$\overline{q}(z) = \overline{q(z)}.$$

7. If $\Gamma : \gamma_1(t)\ (t \in [a,b])$ is a smooth path of length L, then there is an equivalent parametrization $\gamma(s)\ (s \in [0,L])$ such that

$$|\gamma'(s)| = 1, \quad \text{for } 0 \le s \le L.$$

Such an equivalent parametrization is called a **unit-speed parametrization**.

8. If $\Gamma : \gamma(s)\ (s \in [0,L])$ is a smooth path with unit-speed parametrization, which lies in the region $\mathcal{R}$ of a flow velocity function q, then, for each $s \in [0,L]$, the flow velocity $q(\gamma(s))$ has

(a) **tangential component** $q_T(s)$ in the direction specified by $\gamma'(s)$;

(b) **normal component** $q_N(s)$ in the direction specified by $-i\gamma'(s)$.

9. The **circulation** of q along Γ is

$$\mathcal{C}_\Gamma = \int_0^L q_T(s)\,ds,$$

and the **flux** of q across Γ is

$$\mathcal{F}_\Gamma = \int_0^L q_N(s)\,ds.$$

10. $$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \overline{q}(z)\,dz.$$

11. A velocity function with domain a region $\mathcal{R}$ is

(a) **locally circulation-free** if $\mathcal{C}_\Gamma = 0$ for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$;

(b) **locally flux-free** if $\mathcal{F}_\Gamma = 0$ for each simple-closed contour in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

12. A **model flow** of a fluid is described by a continuous complex velocity function (whose domain is a region) which is locally circulation-free and locally flux-free.

13. A fluid flow is a model flow on a region $\mathcal{R}$ if and only if

$$\int_\Gamma \overline{q}(z)\,dz = 0,$$

for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

14. A steady two-dimensional fluid flow with continuous velocity function q on a region $\mathcal{R}$ is a model flow if and only if its conjugate velocity function $\overline{q}$ is analytic on $\mathcal{R}$.

15. Let q be a model flow velocity function with domain a region $\mathcal{R}$, and let D be a punctured open disc in $\mathcal{R}$ with centre α. Then

(a) α is a **source of strength** $\mathcal{F}$ if $\mathcal{F}_\Gamma = \mathcal{F} \neq 0$ for each simple-closed contour Γ in D which surrounds α;

(b) α is a **vortex of strength** $\mathcal{C}$ if $\mathcal{C}_\Gamma = \mathcal{C} \neq 0$ for each simple-closed contour Γ in D which surrounds α.

A source of negative strength is a **sink**.

16. Let q be a continuous complex velocity function on a region $\mathcal{R}$, and suppose that $q_1 = \mathrm{Re}\, q$ and $q_2 = \mathrm{Im}\, q$ have partial derivatives with respect to x and y which are continuous. Then q is

(a) locally circulation-free if and only if

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \text{ on } \mathcal{R};$$

(b) locally flux-free if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \text{ on } \mathcal{R}.$$

Section 2: Complex Potential Functions

1. Let q be a model flow velocity function with domain $\mathcal{R}$. A function Ω which is a primitive of $\overline{q}$ is called a **complex potential function** for the flow.

Moreover,

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \Omega'(z)\,dz = \Omega(\beta) - \Omega(\alpha),$$

where Γ is any contour lying in the domain of Ω with initial and final points α and β.

Such a complex potential function Ω always exists on a simply-connected subregion of $\mathcal{R}$, by the Primitive Theorem.

2. If Ω is a complex potential function on a simply-connected subregion $\mathcal{S}$ of the flow region $\mathcal{R}$, then

$\operatorname{Im}(\Omega(z))$ is constant along each streamline within $\mathcal{S}$.

3. Each point of the flow region has just one streamline through it.

4. If $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$, then $\Psi = \operatorname{Im}\Omega$ is the **stream function** for the model flow with complex potential function Ω and the family of curves given by

$$\Psi(x,y) = \text{constant}$$

forms the steamlines of the flow.

5. There is no flux across a streamline, and so a streamline of given flow can act as a boundary of another flow.

Section 3: Flow past an Obstacle

1. Some standard notation:

$$K_a = \{z : |z| \le a\};$$

$$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic\operatorname{Log} z;$$

$$q_{a,c}(z) = \overline{\Omega'_{a,c}(z)} = \overline{1 - \frac{a^2}{z^2} - \frac{ic}{z}};$$

where $a, c \in \mathbb{R}$, with $a > 0$.

2. For $a > 0$, $c \in \mathbb{R}$, the model flow velocity function $q_{a,c}$ satisfies the following properties:

(a) $\lim\limits_{z\to\infty} q_{a,c}(z) = 1;$

(b) ∂K_a is made up of streamlines for $q_{a,c}$;

(c) for any simple-closed contour surrounding K_a,

(i) $\mathcal{C}_\Gamma = \operatorname{Re}\displaystyle\int_\Gamma \overline{q_{a,c}(z)}\,dz = 2\pi c,$

and

(ii) $\mathcal{F}_\Gamma = \operatorname{Im}\displaystyle\int_\Gamma \overline{q_{a,c}(z)}\,dz = 0.$

3. An **obstacle** is a compact connected set K in $\mathbb{C}$, such that $\mathbb{C} - K$ is also connected.

4. **Obstacle Problem**

Given an obstacle K and a real number c, we seek a model flow velocity function q with domain the region $\mathcal{R} = \mathbb{C} - K$, satisfying the following properties:

(a) $\lim\limits_{z\to\infty} q(z) = 1;$

(b) there is a complex potential function Ω for q on either $\mathcal{R}$ or $\mathcal{R} - \Sigma$, where Σ is a simple smooth path in $\mathcal{R}$, and a real constant k such that

for each $\alpha \in \partial K$, we have $\lim\limits_{z\to\alpha} \operatorname{Im}(\Omega(z)) = k;$

(c) for any simple-closed contour Γ in $\mathcal{R}$ surrounding K,

$\mathcal{C}_\Gamma = 2\pi c.$

5. For the flow in the Obstacle Problem, $2\pi c$ is called the **circulation around the obstacle** K.

6. The model flow velocity function $q_{a,c}$ solves the Obstacle Problem for $K = K_a$, with circulation $2\pi c$ around K.

7. **Flow Mapping Theorem**

Let K be an obstacle and let f be a one-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$ and $a_0, a_{-1}, a_{-2}, \ldots \in \mathbb{C}$. Then the velocity function

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$

is the unique solution to the Obstacle Problem for K, with complex potential function

$$\Omega = \Omega_{a,c} \circ f.$$

8. The function

$$J_a(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $a > 0$, has the inverse function

$$J_a^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4a^2/w^2}\right)$$

$$(w \in \mathbb{C} - [-2a, 2a]).$$

9. A point at which a function f has zero derivative is a **critical point of** f.

10. A **Joukowski aerofoil** (in the z-plane) is an obstacle which (possibly after an appropriate translation or rotation) has boundary $J_a(B)$, where B is a circle which passes through the critical point $w = a$ of the function $J_a(w) = w + a^2/w$ and surrounds the other critical point $w = -a$.

The point $z = 2a$ is called the **trailing edge** of the aerofoil.

11. A symmetric aerofoil in a uniform stream in the direction of the positive x-axis has **angle of attack** ϕ, if ϕ is the angle measured clockwise from the negative x-axis to the line of symmetry of the aerofoil.

12. The restriction of the function

$$J_\alpha(z) = z + \frac{\alpha^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, to $\mathbb{C} - K_{|\alpha|}$ is one-one and has inverse function

$$J_\alpha^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4\alpha^2/w^2}\right) \qquad (w \in \mathbb{C} - L(-2\alpha, 2\alpha)),$$

where $L(-2\alpha, 2\alpha)$ is the line segment between -2α and 2α.

13. The following identity is often useful:

$$J_\alpha^{-1}(w) + \alpha^2/J_\alpha^{-1}(w) = w.$$

Section 4: Lift and Drag

1. In a fluid of constant density ρ, the relationship between fluid pressure $p(z)$ and fluid speed $|q(z)|$ is given by **Bernoulli's Equation**

$$p(z) + \tfrac{1}{2}\rho|q(z)|^2 = p_0,$$

where the constant p_0 is the fluid pressure at any stagnation point.

2. The **total force** F acting on an obstacle K whose boundary is a simple-closed smooth path of length L is

$$F = i\int_0^L p(\gamma(s))\gamma'(s)\,ds,$$

where $\gamma: [0, L] \longrightarrow \partial K$ is a unit-speed parametrization of ∂K.

3. If F is the total force acting on an obstacle, due to a fluid whose velocity function satisfies the Obstacle Problem, then

(a) $\operatorname{Re} F$ is called the **drag** on the obstacle;

(b) $\operatorname{Im} F$ is called the **lift** on the obstacle.

4. Blasius' Theorem

Let q be a solution to the Obstacle Problem for an obstacle K, whose boundary ∂K is a simple-closed smooth path, and suppose that q has a continuous extension to ∂K. Then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho\int_{\partial K} (\overline{q}(z))^2\,dz,$$

where ρ is the density of the fluid.

5. If K is an obstacle whose boundary is not a simple-closed smooth path, then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho\int_{\Gamma} (\overline{q}(z))^2\,dz,$$

where Γ is any simple-closed contour surrounding K.

6. Kutta-Joukowski Lift Theorem

Let q be a solution to the Obstacle Problem for an obstacle K, with circulation $2\pi c$ around K. Then the obstacle experiences a total force

$$F = -2\pi c\rho i.$$

7. Kutta-Joukowski Hypothesis

The circulation around a Joukowski aerofoil is such that the flow velocity q is bounded throughout the flow region, and $q(z)$ tends to a limit as z approaches the trailing edge of the aerofoil.

Unit D3 The Mandelbrot Set

Section 1: Iteration of Analytic Functions

1. A sequence $\{z_n\}$ defined by
$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$
where f is a function, is called an **iteration sequence** with **initial term** z_0.

2. The **nth iterate** of a function f is the function obtained by applying the function f exactly n times:
$$f^n = f \circ f \circ \cdots \circ f.$$
Also, f^0 denotes the identity function $f^0(z) = z$.

3. A point α is a **fixed point** of a function f if
$$f(\alpha) = \alpha.$$
The equation $f(z) = z$ is called the **fixed point equation**.

4. Let α be a fixed point of an analytic function f and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that
$$\lim_{n\to\infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

5. The fixed point α of an analytic function f is
   (a) **attracting**, if $|f'(\alpha)| < 1$;
   (b) **repelling**, if $|f'(\alpha)| > 1$;
   (c) **indifferent**, if $|f'(\alpha)| = 1$;
   (d) **super-attracting**, if $f'(\alpha) = 0$.

6. If α is an attracting fixed point of an analytic function f, then the **basin of attraction** of α under f is the set
$$\{z : f^n(z) \to \alpha \text{ as } n \to \infty\}.$$

7. The functions f and g are **conjugate** to each other if
$$g = h \circ f \circ h^{-1},$$
for some one-one function h called the **conjugating function**. If the sequence $\{z_n\}$ is defined by
$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$
for some z_0, and $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, then the sequence $\{w_n\}$ satisfies
$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \ldots,$$
and $\{z_n\}$ and $\{w_n\}$ are called **conjugate iteration sequences**.

8. Let p be a polynomial function. Then $\{z_n\}$, where
$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \ldots,$$
is the **Newton–Raphson iteration sequence**, and the function N, where
$$N(z) = z - \frac{p(z)}{p'(z)},$$
is the **Newton–Raphson function** corresponding to p.

Section 2: Iterating Complex Quadratics

1. The iteration sequence
$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \ldots,$$
where $a \neq 0$, is conjugate to the iteration sequence
$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$
where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is
$$h(z) = az + \tfrac{1}{2}b.$$

2. The set of functions $\{P_c : c \in \mathbb{C}\}$ defined by
$$P_c(z) = z^2 + c,$$
where $c \in \mathbb{C}$, is the **family of basic quadratic functions**.

3. Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,
$$\{|P_c^n(z_0)|\} \text{ is an increasing sequence,}$$
and
$$P_c^n(z_0) \to \infty \text{ as } n \to \infty.$$

4. For $c \in \mathbb{C}$, the **escape set** E_c is
$$E_c = \{z : P_c^n(z) \to \infty \text{ as } n \to \infty\}.$$
The **keep set** K_c is the complement of E_c.

5. A set A is **completely invariant** under a function f if
$$z \in A \Longleftrightarrow f(z) \in A.$$

6. For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:
   (a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \le r_c\}$;
   (b) E_c is open and K_c is closed;
   (c) $E_c \neq \mathbb{C}$ and $K_c \neq \varnothing$;
   (d) E_c and K_c are each completely invariant under P_c;
   (e) E_c and K_c are each symmetric under rotation by π about 0;
   (f) E_c is connected and K_c has no holes in it.

7. The point α is a **periodic point**, with period p, of a function f if
$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \ldots, p-1.$$
The p points
$$\alpha,\ f(\alpha),\ f^2(\alpha),\ \ldots,\ f^{p-1}(\alpha)$$
then form a **cycle of period p**, or a **p-cycle** of f.
All the periodic points of P_c lie in the keep set K_c.

8. Let α, $f(\alpha)$, $f^2(\alpha)$, $\ldots$, $f^{p-1}(\alpha)$ form a p-cycle of an analytic function f. Then

(a) $$(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'(f^2(\alpha)) \times \cdots \times f'(f^{p-1}(\alpha))$$

and, moreover,

(b) the derivative of f^p takes the same value at each point of the p-cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \ldots = (f^p)'(f^{p-1}(\alpha)).$$

9. Let α belong to a p-cycle of an analytic function f. The number $(f^p)'(\alpha)$ is the **multiplier** of the p-cycle.

10. If α is a periodic point, with period p, of an analytic function f, then α and the corresponding p-cycle are

(a) **attracting**, if $|(f^p)'(\alpha)| < 1$;

(b) **repelling**, if $|(f^p)'(\alpha)| > 1$;

(c) **indifferent**, if $|(f^p)'(\alpha)| = 1$;

(d) **super-attracting**, if $(f^p)'(\alpha) = 0$.

11. Let α be a periodic point of the function P_c.

(a) If α is attracting, then α is an interior point of K_c.

(b) If α is repelling, then α is a boundary point of K_c.

12. The **Julia set** J_c of P_c is the boundary of K_c. K_c is called the 'filled-in Julia set'.

Section 3: Graphical Iteration

1. **Graphical iteration** with a real function f is the process of constructing the sequence $\{x_n\}$, where

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

for a given x_0, by drawing alternately vertical and horizontal lines joining the points

$(x_0, 0)$ to (x_0, x_1) $\quad (= (x_0, f(x_0)))$
(x_0, x_1) to (x_1, x_1)
(x_1, x_1) to (x_1, x_2) $\quad (= (x_1, f(x_1)))$
(x_1, x_2) to (x_2, x_2)
(x_2, x_2) to (x_2, x_3) $\quad (= (x_2, f(x_2)))$
and so on,

using the graphs $y = f(x)$ and $y = x$.

2. If $c \in \mathbb{R}$, then the real function P_c has

(a) no real fixed points if $c > \frac{1}{4}$;

(b) the single fixed point $\frac{1}{2}$, if $c = \frac{1}{4}$;

(c) the two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, if $c < \frac{1}{4}$.

3. If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \varnothing$.

4. If $-2 \le c \le \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$, where

$$I_c = \left[-\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c},\ \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}\right].$$

5. If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c from which a sequence of disjoint, non-empty, open subintervals of I_c has been removed. In particular, $0 \notin K_c$.

Section 4: The Mandelbrot Set

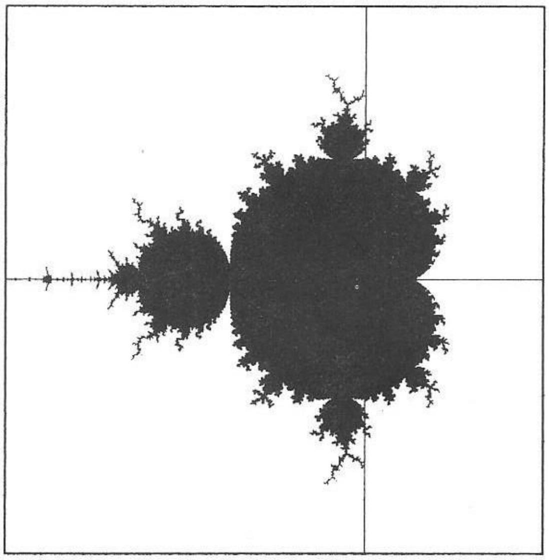
1. A set A is **disconnected** if there are disjoint open sets G_1 and G_2 such that

$$A \cap G_1 \neq \varnothing, \quad A \cap G_2 \neq \varnothing \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$

A set A is **connected** if it is not disconnected.

2. Any pathwise connected set is connected.

3. The **Mandelbrot set** is the set M of complex numbers c such that K_c is connected.



The square represented in this figure is

$$\{c : -2 \le \operatorname{Re} c \le 1, -1.5 \le \operatorname{Im} c \le 1.5\}.$$

4. For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

5. The Mandelbrot set M can be specified as follows:

$$M = \{c : |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots\}.$$

6. The Mandelbrot set M

(a) is a compact subset of $\{c : |c| \le 2\}$;

(b) is symmetric under reflection in the real axis;

(c) meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$;

(d) has no holes in it.

7. The Mandelbrot set is connected.

8. If the function P_c has an attracting cycle, then $c \in M$.

9. (a) The function P_c has an attracting fixed point if and only if c satisfies

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3.$$

(b) The function P_c has an attracting 2-cycle if and only if c satisfies

$$|c + 1| < \tfrac{1}{4}.$$

10. A **periodic region** is a maximal region $\mathcal{R}$ such that, for some positive integer p,

the function P_c has an attracting p-cycle,

for all $c \in \mathcal{R}$.

11. The function P_c has a super-attracting p-cycle if and only if

$$P_c^p(0) = 0, \text{ but } P_c^k(0) \neq 0, \text{ for } k = 1, 2, \ldots, p - 1.$$

12. The number λ is a **primitive nth root of unity** if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$.

13. Suppose that the function P_{c_0}, $c_0 \in \mathbb{C}$, has a p-cycle whose multiplier λ is a root of unity.

(a) **Saddle-node bifurcation at c_0** If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region $\mathcal{R}$, such that

P_c has an attracting p-cycle, for $c \in \mathcal{R}$.

(b) **Period-multiplying bifurcation at c_0** If λ is a primitive nth root of unity, for $n > 1$, then there are two periodic regions $\mathcal{R}_1$ and $\mathcal{R}_2$ whose boundaries meet at c_0 such that

$$P_c \text{ has an attracting } \begin{cases} p\text{-cycle}, & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle}, & \text{for } c \in \mathcal{R}_2. \end{cases}$$

14. The set of points which are mapped to a set E by P_c is called the **preimage set of E** and is denoted by $P_c^{-1}(E)$:

$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

15. A **compact disc** is a compact set whose boundary is a simple-closed smooth path.

16. If E is a compact disc and $c \notin \partial E$, then $P_c^{-1}(E)$ is

(a) one compact disc containing 0, if $c \in \mathrm{int}\,E$;

(b) two compact discs, neither containing 0, if $c \in \mathrm{ext}\,E$.

Section 5: Beyond the Mandelbrot Set

This is a reading-only section.

PART II: INDEX

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